A COLLECTION OF EXAMPLES

ON THE

ANALYTIC GEOMETRY OF PLANE CONICS.

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A COLLECTION OF EXAMPLES

ON THE

ANALYTIC GEOMETRY OF PLANE CONICS;

TO WHICH ARE ADDED

SOME EXAMPLES ON SPHERO-CONICS.

BY

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PREFACE.

A PART of this collection of examples has been published by me before in a *Collection of Examples and Problems on Conics and some of the Higher Plane Curves*. In this volume I have added a good many more examples, besides giving solutions of the more difficult ones which were left unsolved. I believe that either the examples themselves, or the methods of their solution, are to a great extent original.

A large number of the examples contain properties of circles connected with a conic, and especially of those which have double contact with the curve. In proving the properties of the latter systems of circles I have made frequent use of their differential equations in elliptic co-ordinates, the given curve being one of the system of confocal conics. In the same co-ordinates I have also made use of the differential equations of the tangents to a conic, and the systems of conics having double contact with two fixed confocal conics. The method of elliptic co-ordinates simplifies greatly the study of relations involving the angles of intersection of such systems, whose differential equations take a simple form.

I have added a section on Sphero-Conics at the end of the book. Most of these examples are extensions of results already obtained for the case of the plane curves. I have again here made a free use of elliptic co-ordinates.

I have assumed the reader to be familiar with Dr. Salmon's *Conic Sections*, and have constantly made references throughout to that work. I have also occasionally referred to his works on the *Higher Plane Curves*, and *Geometry of Three Dimensions*.

March, *1884*.

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EXAMPLES ON CONICS.

I.—INSCRIBED TRIANGLES.

1. To find the equation of the circle circumscribing a triangle inscribed in a conic.

Let the equation of the conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv U = 0,$$

and that of the circle

$$x^2 + y^2 - 2x'x - 2y'y + k^2 \equiv S = 0\ ;$$

then if P, Q are a pair of chords of intersection of U and S, we must have a relation of the form $S - h^2U = \lambda PQ$, where h and λ are constants. But if P, Q be expressed in terms of the eccentric angles of their extremities, we have (Salmon's *Conics*, Art. 231, Ex. 2),

$$P \equiv \frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta),$$

$$Q \equiv \frac{x}{a}\cos\tfrac{1}{2}(\gamma + \delta) + \frac{y}{b}\sin\tfrac{1}{2}(\gamma + \delta) - \cos\tfrac{1}{2}(\gamma - \delta)\ ;$$

hence, equating the coefficients of x^2, y^2, &c., in the identity $S - h^2U = \lambda PQ$, we get

$$a^2 - h^2 = \lambda\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\gamma + \delta), \qquad (1)$$

$$b^2 - h^2 = \lambda\sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\gamma + \delta), \qquad (2)$$

$$0 = \sin \tfrac{1}{2}(\alpha + \beta + \gamma + \delta), \tag{3}$$

$$2ax' = \lambda\{\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\gamma+\delta)+\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma-\delta)\} \tag{4}$$

$$2by' = \lambda\{\cos\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\gamma+\delta)+\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\gamma-\delta)\} \tag{5}$$

$$k^2 + h^2 = \lambda \cos\tfrac{1}{2}(\alpha - \beta)\cos\tfrac{1}{2}(\gamma - \delta). \tag{6}$$

From (1), (2), (3), we obtain $\alpha + \beta + \gamma + \delta = 0$, $\lambda = a^2 - b^2$, and $h^2 = a^2 \sin^2 \frac{1}{2}(\alpha + \beta) + b^2 \cos^2 \frac{1}{2}(\alpha + \beta)$ = the square of the semi-diameter parallel to P or Q; and substituting these values of δ, λ, and h in (4), (5), (6), we get

$$x' = \frac{a^2 - b^2}{a} \cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\beta + \gamma)\cos\tfrac{1}{2}(\gamma + \alpha), \tag{7}$$

$$y' = \frac{b^2 - a^2}{b} \sin\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\beta + \gamma)\sin\tfrac{1}{2}(\gamma + \alpha), \tag{8}$$

$$k^2 = \tfrac{1}{2}(a^2 - b^2)\{\cos(\alpha + \beta) + \cos(\beta + \gamma) + \cos(\gamma + \alpha)\} - \tfrac{1}{2}(a^2 + b^2). \tag{9}$$

Since the discriminant of $S - h^2U$ vanishes, h^2 must satisfy the equation

$$\frac{x'^2}{a^2 - h^2} + \frac{y'^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0,$$

or
$$h^6 - h^4(a^2 + b^2 + r^2 - x'^2 - y'^2) + h^2\{a^2b^2 + r^2(a^2 + b^2) - b^2x'^2 - a^2y'^2\} - a^2b^2r^2 = 0, \tag{10}$$

where r is the radius of S. Now the three values h_1^2, h_2^2, h_3^2 obtained from this equation are evidently the squares of the semi-diameters parallel to the sides of the triangle; hence from the absolute term of (10) $r = \dfrac{h_1h_2h_3}{ab}$. Also, if we put $a^2 - h^2$ or $b^2 - h^2 = t$ in (10), we can find the expressions for x', y' from the absolute term of the equation in t.

We might find the co-ordinates of the centre of S thus: Eliminating y between the equations of S and U, we get

$$(a^2 - b^2)x^4 - 4a^2x'x^3 + \&c. = 0;$$

hence, if x_1, x_2, x_3, x_4 belong to the four points of intersection of S and U,

$$x' = \frac{a^2 - b^2}{4a^2}(x_1 + x_2 + x_3 + x_4)$$

$$= \frac{a^2 - b^2}{4a}\{\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)\}, \quad (11)$$

which is equivalent to the expression already obtained. Similarly, by eliminating x, we get

$$y' = \frac{b^2 - a^2}{4b}\{\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha + \beta + \gamma)\}. \quad (12)$$

2. To find the equation of the circle through the middle points of the sides of the same triangle.

Let the equation of the circle be

$$x^2 + y^2 - 2x''x - 2y''y + k'^2 = 0,$$

and let x', y' be the co-ordinates of the centre of the circumscribing circle, and a, β those of the centroid; then, by a known geometrical relation,

$$2x'' = 3a - x', \quad 2y'' = 3\beta - y'.$$

Now we have

$$a = \tfrac{1}{3}a(\cos\alpha + \cos\beta + \cos\gamma), \quad \beta = \tfrac{1}{3}b(\sin\alpha + \sin\beta + \sin\gamma);$$

hence we have

$$x'' = \frac{(3a^2 + b^2)}{8a}(\cos\alpha + \cos\beta + \cos\gamma) - \frac{(a^2 - b^2)}{8a}\cos(\alpha + \beta + \gamma), \quad (1)$$

$$y'' = \frac{(b^2 + 3a^2)}{8b}(\sin\alpha + \sin\beta + \sin\gamma) + \frac{(a^2 - b^2)}{8b}\sin(\alpha + \beta + \gamma). \quad (2)$$

To find k'^2 we have $k'^2 = x''^2 + y''^2 - \frac{1}{4}r^2$, where r is the radius of the circumscribed circle; hence

$$\begin{aligned} k'^2 &= \tfrac{1}{4}\{(3\alpha - x')^2 + (3\beta - y')^2 - r^2\} \\ &= \tfrac{1}{4}\{3\alpha(3\alpha - 2x') + 3\beta(3\beta - 2y') + x'^2 + y'^2 - r^2\} \\ &= \tfrac{1}{8}(\cos\alpha + \cos\beta + \cos\gamma)\{(a^2+b^2)(\cos\alpha + \cos\beta + \cos\gamma) \\ &\qquad\qquad - (a^2 - b^2)\cos(\alpha+\beta+\gamma)\} \\ &\quad + \tfrac{1}{8}(\sin\alpha + \sin\beta + \sin\gamma)\{(a^2+b^2)(\sin\alpha + \sin\beta + \sin\gamma) \\ &\qquad\qquad - (a^2 - b^2)\sin(\alpha+\beta+\gamma)\} \\ &\quad + \tfrac{1}{8}(a^2 - b^2)\{\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha)\} - \tfrac{1}{8}(a^2+b^2) \\ &= \tfrac{1}{8}(a^2+b^2)\{(\cos\alpha + \cos\beta \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 - 1\} \\ &= \tfrac{1}{4}(a^2+b^2)\{1 + \cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha)\} \\ &= (a^2+b^2)\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha). \qquad (3) \end{aligned}$$

3. To find the locus of the centroid of an equilateral triangle inscribed in a conic.

Equating the co-ordinates of the centroid and the centre of the circumscribing circle, we get, if $\alpha + \beta + \gamma = -\delta$,

$$x = \frac{a(a^2 - b^2)\cos\delta}{a^2 + 3b^2}, \quad y = -\frac{b(a^2 - b^2)\sin\delta}{b^2 + 3a^2};$$

hence the locus is the conic

$$\frac{x^2}{a^2}(a^2 + 3b^2)^2 + \frac{y^2}{b^2}(b^2 + 3a^2)^2 = (a^2 - b^2)^2.$$

We may find the locus thus for the parabola $y^2 - px = 0$. Let $x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$ be the equation of the circumscribing circle, then eliminating y between the equations of the circle and parabola, we get $x^4 + 2(p - 2\alpha)x^3 + \&c. = 0$, whence $2(2\alpha - p) = x_1 + x_2 + x_3 + x_4$; and similarly eliminating x, we find $y_1 + y_2 + y_3 + y_4 = 0$. But the centre of the circle being the centroid of the triangle, we have $x_1+x_2+x_3=3\alpha$, $y_1 + y_2 + y_3 = 3\beta$; hence $\alpha = 2p + x_4$, $3\beta = -y_4$; therefore $9\beta^2 = p(\alpha - 2p)$.

4. If the centroid of a triangle inscribed in a hyperbola lies on the curve, the eccentric angles must satisfy the equation $(\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 9$, or $\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha) = 1$ (1); hence from Ex. 2 (3), we have $k'^2 = a^2 + b^2$, from which we see that the circle passing through the middle points of the sides of the triangle cuts orthogonally the director circle. The eccentric angles are of course imaginary for the hyperbola, but this does not affect the validity of the proof. When the relation (1) is satisfied, the area of the triangle formed by the tangents at the vertices is equal to half the area of the given triangle.

The relation (1) can also be written in the form

$$\sqrt{\sin(\beta - \gamma)} + \sqrt{\sin(\gamma - \alpha)} + \sqrt{\sin(\alpha - \beta)} = 0,$$

from which it can be seen that the ellipse touching the sides of the triangle at their middle points passes through the centre of the curve.

5. If
$$x^2 + y^2 - 2x'x - 2y'y + k^2 = 0 \qquad (1)$$

represents the circle passing through the extremities of three semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, show that

$$x^2 + y^2 - \frac{2b}{a}y'x + \frac{2a}{b}x'y - (a^2 + b^2 + k^2) = 0 \qquad (2)$$

represents the circle passing through the extremities of the three conjugate semi-diameters.

Also show that if the circle (1) cuts orthogonally the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k'^2 = 0,$$

the circle (2) cuts orthogonally

$$x^2 + y^2 + \frac{2a}{b}\beta x - \frac{2b}{a}\alpha y + a^2 + b^2 - k'^2 = 0.$$

6. Corresponding points on the confocal conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv U = 0, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 \equiv V = 0$$

are connected by the relations

$$\frac{x}{a} = \frac{x'}{a'}, \quad \frac{y}{b} = \frac{y'}{b'};$$

show that if $x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$ represents the circle passing through three points on U,

$$x^2 + y^2 - \frac{2a}{a'}\alpha x - \frac{2b}{b'}\beta y + k^2 + a^2 - a'^2 = 0$$

will represent the circle passing through the three corresponding points on V.

7. To find the locus of the centroid of a triangle inscribed in one conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv U = 0$, and circumscribed about another whose tangential equation is

$$(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 \equiv \Sigma = 0.$$

Writing down the condition that the chord of U,

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta) = 0,$$

should touch Σ, and two similar equations for the other sides of the triangle, multiplying them by $\sin(\alpha - \beta)$, $\sin(\beta - \gamma)$,

$\sin(\gamma - \alpha)$, respectively, and adding them together we get, after dividing by $\sin\frac{1}{2}(\alpha - \beta)$, $\sin\frac{1}{2}(\beta - \gamma)$, $\sin\frac{1}{2}(\gamma - \alpha)$,

$$C\{1 + 4\cos\tfrac{1}{2}(\alpha - \beta)\cos\tfrac{1}{2}(\beta - \gamma)\cos\tfrac{1}{2}(\gamma - \alpha)\}$$
$$- 2\frac{G}{a}(\cos\alpha + \cos\beta + \cos\gamma)$$
$$- 2\frac{F}{b}(\sin\alpha + \sin\beta + \sin\gamma) + \frac{A}{a^2} + \frac{B}{b^2} = 0. \qquad (1)$$

But $\cos\alpha + \cos\beta + \cos\gamma = \dfrac{3x}{a}$, $\sin\alpha + \sin\beta + \sin\gamma = 3\dfrac{y}{b}$,

and $1 + 8\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha) = 9\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right)$;

hence, from (1), we have for the equation of the locus the conic

$$9C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - 12\left(\frac{Gx}{a^2} + \frac{Fy}{b^2}\right) + 2\left(\frac{A}{a^2} + \frac{B}{b^2}\right) + C' = 0. \quad (2)$$

There is, of course, an invariant relation connecting Σ and U; and if we have also $\dfrac{A}{a^2} = \dfrac{B}{b^2}$, $H = 0$, it can be shown that the centroid of the triangle is fixed. For, if we eliminate γ between the equations

$$\cos\alpha + \cos\beta + \cos\gamma = \frac{3x'}{a}, \ \sin\alpha + \sin\beta + \sin\gamma = \frac{3y'}{b},$$

we get

$$9\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)1 + 4\cos^2\tfrac{1}{2}(\alpha - \beta) - 12\frac{x'}{a}\cos\tfrac{1}{2}(\alpha - \beta)\cos\tfrac{1}{2}(\alpha + \beta)$$
$$- 12\frac{y'}{b}\cos\tfrac{1}{2}(\alpha - \beta)\sin\tfrac{1}{2}(\alpha + \beta) = 0,$$

or $\left(\dfrac{9x'^2}{a^2} + \dfrac{9y'^2}{b^2} - 1\right)(a^2\lambda^2 + b^2\mu^2) + 4\nu^2 + 12\dfrac{x'}{a}\nu\lambda + 12\dfrac{y'}{b}\mu\nu = 0.$

Putting $C = 0$ in (2), we see that the locus of the centroid of a triangle circumscribed about a conic and inscribed in a parabola is a right line. This result may be readily arrived at geometrically by projecting the conic orthogonally into a circle. The centre of the circumscribed circle is then fixed, and the intersection of perpendiculars lies on the directrix of the parabola; therefore, &c.

8. Since the centroid of a triangle is the pole of the line at infinity with regard to the triangle, by projecting the results of the preceding example we see that, if a triangle be inscribed in a conic U, and circumscribed about a conic V, the locus of the pole of a fixed line L, with regard to the triangle, is a conic, which reduces to a line if L touches V, the other factor being L. Also, if U, V, and L are connected by two relations besides the invariant relation between U and V, the pole of L, with regard to the triangle, is fixed. L is then, in fact, the chord of contact of two tangents of U which touch V.

9. A, B, C are the vertices of a fixed triangle inscribed in a conic, and P is a variable point on the curve: to find the locus of the centroid of the triangle formed by the lines bisecting PA, PB, PC, at right angles.

Let the conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and α, β, γ, ϕ the eccentric angles of A, B, C, P, respectively; then the intersection (x_1, y_1) of the lines bisecting PA, PB at right angles is the centre of the circle passing through P, A, B; hence, from (11) and (12) Ex. 1, we have

$$x_1 = \frac{c^2}{4a}\{\cos\alpha + \cos\beta + \cos\phi + \cos(\alpha + \beta + \phi)\},$$

$$y_1 = -\frac{c^2}{4b}\{\sin\alpha + \sin\beta + \sin\phi - \sin(\alpha + \beta + \phi)\},$$

and similar values for the co-ordinates $x_2 y_2$, $x_3 y_3$ of the other vertices of the variable triangle.

But $x = \frac{1}{3}(x_1 + x_2 + x_3) = \frac{c^2}{12a}\{2(\cos\alpha + \cos\beta + \cos\gamma)$

$+ 3\cos\phi + \cos(\alpha+\beta+\phi) + \cos(\beta+\gamma+\phi) + \cos(\gamma+\alpha+\phi)\}$,

and $y = \frac{1}{3}(y_1 + y_2 + y_3) = -\frac{c^2}{12b}\{2(\sin\alpha + \sin\beta + \sin\gamma)$

$+ 3\sin\phi - \sin(\alpha+\beta+\phi) - \sin(\beta+\gamma+\phi) - \sin(\gamma+\alpha+\phi)\}$,

where x, y are the co-ordinates of the centroid of the variable triangle; hence, eliminating ϕ between these equations, we see that the locus is in general a conic. When the centroid of the fixed triangle coincides with the centre of the curve, we have $\cos\alpha + \cos\beta + \cos\gamma = 0$, $\sin\alpha + \sin\beta + \sin\gamma = 0$, and the locus is then $a^2x^2 + b^2y^2 = \frac{c^4}{16}$, which is independent of the position of the fixed triangle.

If $(\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 9$, in which case the centroid of the fixed triangle is on the curve, and the conic must be a hyperbola, the locus reduces to a line passing through the centre of the circle circumscribing the fixed triangle.

10. Triangles are inscribed in a conic, so that the circumscribing circles pass through a focus; to show that one of the circles touching the sides has double contact with a fixed conic.

Let A, B, C be the vertices of the triangle, and F a focus of the conic; then, from Ptolemy's theorem, we have

$$BC \,.\, FA \pm CA \,.\, FB \pm AB \,.\, FC = 0. \qquad (1)$$

But FA, FB, FC are proportional to the perpendiculars α, β, γ from A, B, C on the directrix; hence from (1) $BC \,.\, \alpha \pm CA \,.\, \beta \pm AB \,.\, \gamma = 0$ (2), from which it readily follows that the centre of one of the circles touching the sides of the triangle lies on the directrix.

Now when a circle $(x-x')^2+(y-y')^2 \equiv r^2 \equiv S' = 0$ is inscribed in a triangle inscribed in the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 \equiv S = 0$, we must have the relation $\Theta'^2 - 4\Delta'\Theta = 0$ between the invariants, or (Salmon's *Conics*, Art. 371, Ex. 4),

$$\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1-r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}^2 = \frac{4r^2}{a^2b^2}\,(a^2+b^2+r^2-x'^2-y'^2).$$

Solving this equation for r we get

$$r = \frac{ab}{c^2}\left\{\sqrt{\left(a^2-\frac{c^2x'^2}{a^2}\right)} \pm \sqrt{\left(b^2+\frac{c^2y'^2}{b^2}\right)}\right\}, \qquad (3)$$

where $c^2 = a^2 - b^2$; hence, putting $x' = \frac{a^2}{c}$,

$$r^2 = \frac{a^2}{c^2}\left\{y'^2 + \frac{b^4}{c^2}\right\},$$

subject to which conditions it can easily be seen that S' has double contact with the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{c}+\frac{a^2c^2-b^4}{c^4} = 0.$$

11. Triangles are inscribed in a conic and circumscribed about a fixed circle; to find the envelope of the circumscribing circles.

Let A, B, C be the vertices of the triangle, and α, β, γ the perpendiculars from A, B, C on a line drawn through the centre of the fixed circle parallel to one of the axes of the curve, then $BC.\alpha \pm CA.\beta \pm AB.\gamma = 0$ (1). But α, β, γ are proportional to the tangents t_1, t_2, t_3 drawn from A, B, C to the circle having double contact with the curve at the points where it is met by the parallel to the axis (Salmon's *Conics*, Art. 261); hence from (1) $BC.t_1 \pm CA.t_2 \pm AB.t_3 = 0$ (2).

Now, by Dr. Casey's theorem, if four circles are all touched by a fifth, their common tangents (12), &c. are connected by the relation (12) (34) ± (13) (24) ± (23 (14) = 0 (Salmon's *Conics*, Art. 121 (*a*)) ; hence, by supposing the radii of the circles (1), (2), (3) to vanish, we see that the relation (2) expresses that the circle having double contact with the curve touches the circle passing through *A*, *B*, *C*. Since we may draw a parallel to either axis through the centre of the fixed circle, we see that the circumscribing circle will touch the two circles having double contact with the curve at the points where it is met by these parallels.

12. A triangle is inscribed in a conic so that the circumscribing circle touches a fixed circle having double contact with the conic; show that one of the circles touching the sides has double contact with a fixed parallel curve to a conic (see Ex. 11 and Ex. 10 (3)).

13. A triangle, self-conjugate with regard to a conic *V*, is inscribed in a conic *U*; show that circles having double contact with *U*, whose chords of contact touch *V*, cut orthogonally the director circle of *V*.

14. Given a triangle inscribed in a conic, if a focus lies on a fixed circle, to find the envelope of the corresponding directrix.

If ρ_1, ρ_2, ρ_3 denote the distances of a point from the vertices of the triangle, and *a*, *b*, *c* the sides, it can be seen from Ptolemy's theorem that

$$a^4\rho_1^4 + b^4\rho_2^4 + c^4\rho_3^4 - 2b^2c^2\rho_2^2\rho_3^2 - 2c^2a^2\rho_3^2\rho_1^2 - 2a^2b^2\rho_1^2\rho_2^2 \equiv U$$

is proportional to the square of the tangent from the point to the circumscribing circle. It follows then that any circle can be written in the form $U - (l\rho_1^2 + m\rho_2^2 + n\rho_3^2)^2 = 0$ (1). But ρ_1, ρ_2, ρ_3 are proportional to the perpendiculars from the vertices of the triangle on the directrix; hence from (1) we have for the tangential equation of the envelope of the directrix

$$a^4\alpha^4 + b^4\beta^4 + c^4\gamma^4 - 2b^2c^2\beta^2\gamma^2 - 2c^2a^2\gamma^2\alpha^2 - 2a^2b^2\alpha^2\beta^2 - (l\alpha^2 + m\beta^2 + n\gamma^2)^2 = 0.$$

Since $a^4\alpha^4 +$ &c. is the product of the factors

$$a\alpha + b\beta + c\gamma,\ a\alpha + b\beta - c\gamma,\ a\alpha - b\beta + c\gamma,\ a\alpha - b\beta - c\gamma,$$

it follows that the envelope has the centres of the circles touching the sides for double points.

15. Let ABC be a triangle inscribed in a conic U, and Σ a circle having double contact with U at points on a parallel to its minor axis. Let t, t' be the lengths of the direct and transverse common tangents of Σ and the circle circumscribing ABC, and let p_1, p_2, p_3, p_4 be the perpendiculars from the centres of the four circles touching the sides BC, CA, AB on the chord of contact of Σ and U. Show that $t^2 t'^2 = e^4 p_1 p_2 p_3 p_4$, where e is the eccentricity of U (see Ex. 11).

16. A triangle is inscribed in a variable parabola; to find the locus of the intersection of the perpendiculars of the triangle formed by the tangents to the curve at the vertices of the triangle.

Consider two consecutive parabolas indefinitely near one another, circumscribed about the triangle, then the intersections of the perpendiculars of the triangles formed by the tangents lie on the directrices of the corresponding parabolas, and, therefore, the ultimate intersection of the directrices is the point whose locus we seek. Thus we see that the locus coincides with the envelope of the directrix.

Let ρ_1, ρ_2, ρ_3 denote the distances of a point from the vertices of the triangle, and a, b, c, A, B, C the sides and angles, respectively, then we can verify the following identical relation:

$$a^2\rho_1^4 + b^2\rho_2^4 + c^2\rho_3^4 - 2bc\cos A\,\rho_2^2\rho_3^2 - 2ca\cos B\,\rho_3^2\rho_1^2$$
$$- 2ab\cos C\,\rho_1^2\rho_2^2 - 2abc\,(a\cos A\,\rho_1^2 + b\cos B\,\rho_2^2$$
$$+ c\cos C\,\rho_3^2) + a^2b^2c^2 = 0. \qquad (1)$$

But ρ_1, ρ_2, ρ_3 are equal to the perpendiculars α, β, γ from the vertices on the directrix; hence from (1)

$$a^2\alpha^4 + b^2\beta^4 + c^2\gamma^4 - 2bc\cos A\,\beta^2\gamma^2 - 2ca\cos B\,\gamma^2\alpha^2$$
$$- 2ab\cos C\,\alpha^2\beta^2 - 2abc\,(a\cos A\alpha^2 + b\cos B\beta^2$$
$$+ c\cos C\gamma^2) + a^2b^2c^2 = 0, \qquad (2)$$

which being rendered homogeneous by means of the identical relation connecting α, β, γ, viz.,

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - 2bc\cos A\,\beta\gamma - 2ca\cos B\gamma\alpha$$
$$- 2ab\cos C\,\alpha\beta = 4\,\Delta^2,$$

shows that the envelope is of the fourth class.

It can be shown that the curve (2) is universal. For the tangential equation $l\sqrt{\alpha} + m\sqrt{\beta} + n\sqrt{\gamma} = 0$, where $l + m + n = 0$ represents a series of parabolas circumscribing the triangle, and the tangential co-ordinates of the directrix can then be expressed as functions of l, m, n of the fourth degree (Salmon's *Conics*, Art. 383). The locus thus appears to be of the sixth degree.

17. The co-ordinates of the centroid of a triangle ABC, inscribed in the parabola $y^2 - px = 0$, are α, β; show that the co-ordinates x, y of the centroid of the triangle formed by the tangents at A, B, C are given by the equations

$$2px = 3\beta^2 - p\alpha, \quad y = \beta.$$

18. Show that the equation of the circle passing through the middle points of the sides of a triangle, inscribed in the parabola $y^2 - px = 0$, is

$$4p^2(x^2 + y^2) - 2p(p_1^2 - 3p_2 - p^2)x - 2(2p^2 p_1 + p_1 p_2 - p_3)y + 2p_2(p_2 + p^2) = 0,$$

where p_1, p_2, p_3 are the sum, sum of the products in pairs, and product, respectively, of the ordinates y_1 y_2, y_3 of the vertices of the triangle.

19. A triangle ABC is inscribed in a parabola whose focus is F; to show that one of the circles touching the lines which bisect FA, FB, FC at right angles passes through the centre of the circle circumscribing ABC.

Let ρ_1, r_1, r_2, r_3 denote the distances of a point P from F, A, B, C respectively; then if α, β, γ are the perpendiculars from P on the lines bisecting FA, FB, FC at right angles, we have

$$\rho^2 - r_1^2 = 2d_1\alpha, \quad \rho^2 - r_2^2 = 2d_2\beta, \quad \rho^2 - r_3^2 = 2d_3\gamma, \qquad (1)$$

where $d_1 = FA$, &c. Now the equation of a circle touching α, β, γ is

$$\cos\tfrac{1}{2}A\sqrt{\alpha} + \cos\tfrac{1}{2}B\sqrt{\beta} + \cos\tfrac{1}{2}C\sqrt{\gamma} = 0;$$

or from (1)

$$\sin\tfrac{1}{2}(\theta_2 - \theta_3)\sqrt{\left(\frac{\rho^2 - r_1^2}{d_1}\right)} + \sin\tfrac{1}{2}(\theta_3 - \theta_1)\sqrt{\left(\frac{\rho^2 - r_2^2}{d_2}\right)} + \sin\tfrac{1}{2}(\theta_1 - \theta_2)\sqrt{\left(\frac{\rho^2 - r_2^2}{d_3}\right)} = 0, \qquad (2)$$

where θ_1, θ_2, θ_3 are the angles which FA, FB, FC make with the axis of the curve.

But for the centre of the circle passing through A, B, C, $r_1 = r_2 = r_3$, and from the polar equation of the parabola

$$d_1 = m \sec^2 \tfrac{1}{2}\theta_1, \quad d_2 = m \sec^2 \tfrac{1}{2}\theta_2, \quad d_3 = m \sec^2 \tfrac{1}{2}\theta_3,$$

which values satisfy the equation (2) identically; therefore, &c.

20. A triangle is inscribed in the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; if x, y are the co-ordinates of the centre of the circumscribing circle, and x', y' those of the centroid, show that

$$16(a^2x^2 + b^2y^2) + 9c^4\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) - 12c^2(xx' - yy') - c^4 = 0.$$

21. A triangle is inscribed in a conic so that the circumscribing circle passes through two fixed points, one of which is on the curve; show that the centroid and intersection of the perpendiculars lie on fixed lines.

Show also, in the same case, that the nine-point and polar circles have double contact with fixed conics.

22. A triangle is inscribed in a conic U so that two sides are parallel to fixed lines; show that the locus of the centre of the circumscribing circle is a conic having the same centre and axes as U.

23. If the centroid of a triangle inscribed in a conic lies on a concentric, similar and similarly situated conic, show that the nine-point circle cuts orthogonally a fixed circle concentric with the curve. Show also that the area of the triangle is in a constant ratio to the area of the triangle formed by the tangents at the vertices.

24. A, B, C are three points forming a triangle inscribed in a conic, so that the tangent at each vertex is parallel to the opposite side; and P is any point on the curve. If PA, PB, PC meet the opposite sides of the triangle in L, M, N, show

that the area of the triangle LMN is double that of the triangle ABC.

25. An ellipse circumscribes a fixed triangle so that two of the vertices are at the extremities of a pair of conjugate diameters; show that the locus of its centre is a hyperbola with regard to which the given triangle is self-conjugate.

26. A conic circumscribes a fixed triangle ABC; if the diameter of the conic parallel to AB be given in length, show that the locus of its centre is a conic, whose asymptotes are parallel to AC, BC, and with regard to which C is the pole of AB.

27. A conic circumscribes a fixed triangle so that one of the vertices of the triangle is a vertex of the curve; show that the trilinear equation of the locus of the centre of the conic is

$$\alpha(\alpha + \beta \cos C)(\beta \sin B + \gamma \sin C - \alpha \sin A)$$
$$- \beta(\beta + \alpha \cos C)(\alpha \sin A - \beta \sin B + \gamma \sin C) = 0.$$

28. A circle S' circumscribes a triangle ABC inscribed in a conic S; if $\Theta^2 - 4\Delta\Theta' = 0$, show that the algebraic sum of the diameters of S parallel to AB, BC, CA is equal to zero, and if $\Theta'^2 - 4\Delta\Theta' = 0$, show that the algebraic sum of the reciprocals of the diameters is equal to zero, where Δ, Θ, Θ', Δ' are the invariants of S and S' (see Salmon's *Conics*, Art. 371, Ex. 4).

II.—CIRCUMSCRIBED TRIANGLES.

29. To find the equation of the circle circumscribing the triangle formed by three tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Let $S \equiv x^2 + y^2 - 2x'x - 2y'y + k^2 = 0$ be the equation of the circle; then, substituting the co-ordinates

$$\frac{a \cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}, \quad \frac{b \sin \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}$$

of the intersection of the tangents

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha - 1 = 0, \quad \frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta - 1 = 0,$$

in $S = 0$, we get

$$a^2 \cos^2 \tfrac{1}{2}(\alpha + \beta) + b^2 \sin^2 \tfrac{1}{2}(\alpha + \beta) - ax'(\cos \alpha + \cos \beta)$$
$$- by'(\sin \alpha + \sin \beta) + k^2 \cos^2 \tfrac{1}{2}(\alpha - \beta) = 0,$$

or $$a^2 + b^2 + k^2 + (a^2 - b^2) \cos (\alpha + \beta) + k^2 \cos(\alpha - \beta)$$
$$- 2ax'(\cos \alpha + \cos \beta) - 2by'(\sin \alpha + \sin \beta) = 0. \quad (1)$$

Now let $\cos \alpha + \cos \beta + \cos \gamma = p$, $\sin \alpha + \sin \beta + \sin \gamma = q$, $\alpha + \beta + \gamma = \phi$, then we have

$$\cos (\alpha + \beta) = \cos(\phi - \gamma), \quad \cos (\alpha - \beta) = \tfrac{1}{2}(p^2 + q^2 - 1)$$
$$- (p \cos \gamma + q \sin \gamma),$$

and (1) may be written

$$L \cos \gamma + M \sin \gamma + N = 0, \qquad (2)$$

where $L = (a^2 - b^2)\cos\phi + 2ax' - k^2 p,$ (3)

$$M = (a^2 - b^2)\sin\phi + 2by' - k^2 q, \quad (4)$$

$$N = a^2 + b^2 - 2pax' - 2qby' + \tfrac{1}{2}(p^2 + q^2 + 1). \quad (5)$$

But since (2) is true when γ is replaced by α and β, respectively, we must have $L = M = N = 0$. Eliminating, then, x', y' from these equations, we get

$$k^2 = \frac{2\{a^2 + b^2 + (a^2 - b^2)(p\cos\phi + q\sin\phi)\}}{p^2 + q^2 - 1}$$

$$= \frac{a^2 + b^2 + (a^2 - b^2)\{\cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha)\}}{4\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)}, \quad (6)$$

since $p\cos\phi + q\sin\phi = \cos(\alpha+\beta) + \cos(\beta+\gamma) + \cos(\gamma+\alpha)$,

and $p^2 + q^2 - 1 = 4\{1 - \cos^2\frac{1}{2}(\alpha-\beta) - \cos^2\frac{1}{2}(\beta-\gamma) - \cos^2\frac{1}{2}(\gamma-\alpha)\}$

$$= 8\cos\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\gamma-\alpha).$$

Hence, finally, we may write the equation of S thus:

$$x^2 + y^2 - \frac{x}{a}\{k^2(\cos\alpha + \cos\beta + \cos\gamma) + (a^2 - b^2)\cos(\alpha+\beta+\gamma)\}$$

$$-\frac{y}{b}\{k^2(\sin\alpha + \sin\beta + \sin\gamma) + (a^2 - b^2)\sin(\alpha+\beta+\gamma)\}$$

$$+ k^2 = 0, \quad (7)$$

where k^2 has the value (6).

We may also find the equation of the circle as follows:—If R is the radius of the circle, and Δ, A, B, C the area and angles of the triangle, respectively, we have

$$R^2 = \frac{\Delta}{2\sin A\sin B\sin C};$$

but $$\Delta = ab \tan \tfrac{1}{2}(\alpha - \beta) \tan \tfrac{1}{2}(\beta - \gamma) \tan \tfrac{1}{2}(\gamma - \alpha)$$
(Salmon's *Conics*, Art. 231, Ex. 9),

and $$\sin A = \frac{ab \sin (\beta - \gamma)}{\sqrt{\{(a^2 \sin^2\beta + b^2 \cos^2\beta)(a^2 \sin^2 \gamma + b^2 \cos^2 \gamma)\}}},$$
$$\sin B = \&c.;$$

hence we have

$$R = \frac{\sqrt{\{(a^2 \sin^2 \alpha + b^2 \cos^2\alpha)(a^2 \sin^2\beta + b^2 \cos^2\beta)(a^2 \sin^2 \gamma + b^2 \cos^2 \gamma)\}}}{4ab \cos \frac{1}{2}(\alpha - \beta) \cos \frac{1}{2}(\beta - \gamma) \cos \frac{1}{2}(\gamma - \alpha)}. \quad (8)$$

Now let $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ be the perpendiculars from the foci on the sides of the triangle, then $\alpha\alpha' = \beta\beta' = \gamma\gamma' = b^2$, and $\alpha' \sin A + \beta' \sin B + \gamma' \sin C = \frac{\Delta}{R}$, identically; therefore

$$b^2 (\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C) = \frac{\Delta}{R} \alpha\beta\gamma;$$

but $\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = - \sin A \sin B \sin C t^2$
(Salmon's *Conics*, Art. 132, Ex. 2);

hence $$b^2 t^2 = - 2R\alpha\beta\gamma; \quad (9)$$

and, of course, also for the other focus. $$b^2 t'^2 = - 2R\alpha'\beta'\gamma' \quad (10)$$

Now we have $$k^2 - 2cx' + c^2 = t^2,$$
$$k^2 + 2cx' + c^2 = t'^2;$$

and if we express α, β, γ in terms of the eccentric angles, we get

$$\alpha = \frac{b(c \cos \alpha - a)}{\sqrt{\{(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)\}}}, \quad \beta = \&c.;$$

hence (9) and (10) become

$$k^2 - 2cx' + c^2 = \frac{(a - c\cos\alpha)(a - c\cos\beta)(a - c\cos\gamma)}{2a\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)}, \quad (11)$$

$$k^2 + 2cx' + c^2 = \frac{(a + c\cos\alpha)(a + c\cos\beta)(a + c\cos\gamma)}{2a\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)}, \quad (12)$$

which give by subtraction

$$x' = \frac{a^2(\cos\alpha + \cos\beta + \cos\gamma) + c^2\cos\alpha\cos\beta\cos\gamma)}{4a\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)}, \quad (13)$$

which is equivalent to the value already obtained. By symmetry, then, or by means of the imaginary foci, we arrive at

$$y' = \frac{b^2(\sin\alpha + \sin\beta + \sin\gamma) - c^2\sin\alpha\sin\beta\sin\gamma}{4b\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)}. \quad (14)$$

Also adding the equations (11) and (12) we get

$$k^2 = \frac{a^2 + c^2(\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha)}{2\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)} - c^2.$$

From (9) and (10) we get at once

$$t^2 t'^2 = 4b^2R^2, \quad (15)$$

the invariant relation connecting the circle and the conic.

We can obtain an expression for the radius of the circle in terms of the distances ρ_1, ρ_2, ρ_3, ρ_1', ρ_2', ρ_3' of the foci from the vertices of the triangle. Let a', b', c', Δ', be the sides and area respectively of the triangle formed by the feet of the perpendiculars from a focus on the sides of the given triangle; then, since the auxiliary circle of the conic passes through the feet of the perpendiculars, we have

$$a'b'c' = 4a\Delta';$$

but $a' = \rho_1 \sin A,\ b' = \rho_2 \sin B,\ c' = \rho_3 \sin C,$

and $$2\Delta' = -\sin A \sin B \sin C t^2$$

(Salmon's *Conics*, Art. 125).

Hence $$\rho_1 \rho_2 \rho_3 = -2at^2, \qquad (16)$$

and similarly for the other focus,

$$\rho_1' \rho_2' \rho_3' = -2at'^2;$$

therefore from (15)

$$R^2 = \frac{\rho_1 \rho_2 \rho_3 \rho_1' \rho_2' \rho_3'}{16 a^2 b^2}.$$

30. To find the equation of the polar circle of the triangle formed by three tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Let the equation of the circle be $(x - x'')^2 + (y - y'')^2 - r^2 = 0$, and let x', y' be the co-ordinates of the centre of the circumscribing circle, and α, β those of the centroid; then, by a known geometrical relation, we have

$$x'' = 3\alpha - 2x', \quad y'' = 3\beta - 2y'.$$

But $$\frac{3\alpha}{a} = \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)} + \frac{\cos \frac{1}{2}(\beta + \gamma)}{\cos \frac{1}{2}(\beta - \gamma)} + \frac{\cos \frac{1}{2}(\gamma + \alpha)}{\cos \frac{1}{2}(\gamma - \alpha)}$$

$$= \frac{3(\cos\alpha + \cos\beta + \cos\gamma) + \cos(\alpha + \beta + \gamma) + 4\cos\alpha\cos\beta\cos\gamma}{4\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha)};$$

hence, substituting for x' from Ex. 29 (13), we get

$$x'' = \frac{a^2\{\cos\alpha + \cos\beta + \cos\gamma - \cos(\alpha + \beta + \gamma)\} + 2(a^2 + b^2)\cos\alpha\cos\beta\cos\gamma}{4a\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha)};$$

and, similarly,

$$y'' = \frac{b^2\{\sin\alpha + \sin\beta + \sin\gamma + \sin(\alpha + \beta + \gamma)\} + 2(a^2 + b^2)\sin\alpha\sin\beta\sin\gamma}{4b\cos\frac{1}{2}(\alpha - \beta)\cos\frac{1}{2}(\beta - \gamma)\cos\frac{1}{2}(\gamma - \alpha)}.$$

Also the absolute term $x''^2 + y''^2 - r^2 = a^2 + b^2$; for by the invariants we know that this circle cuts orthogonally the director circle (Salmon's *Conics*, Art. 375, Ex. 4).

If r be the radius of this circle, and t_1, t_2, t_3 the lengths of the tangents from the vertices of the triangle to the director circle, we have

$$r^2 = -2\Delta \cot A \cot B \cot C;$$

but $\cot A = \dfrac{t_1^2}{2ab \tan \frac{1}{2}(\beta - \gamma)}$, $\cot B = \&c.$

(Salmon's *Conics*, Art. 169, Ex. 3);

and $\Delta = ab \tan \frac{1}{2}(\beta - \gamma) \tan \frac{1}{2}(\gamma - \alpha) \tan \frac{1}{2}(\alpha - \beta)$;

therefore $r^2 = -\dfrac{t_1^2 t_2^2 t_3^2}{4a^2 b^2}$.

For the ellipse, if r is real, one of the quantities t_1, t_2, t_3 is, of course, imaginary.

For the parabola, $y^2 - 4mx = 0$, we find

$$r^2 = -\frac{p_1 p_2 p_3}{m},$$

where p_1, p_2, p_3 are the perpendiculars from the vertices of the triangle on the directrix.

31. The circle circumscribing the triangle formed by three tangents to a conic passes through the centre of the curve; show that three extremities of the diameters parallel to the tangents lie on a circle passing through the centre of the curve.

32. If the nine-point circle of the triangle formed by three tangents to a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ passes through the centre of the curve, show that the centre of the circumscribing circle lies on the confocal hyperbola $\dfrac{x^2}{b^2} - \dfrac{y^2}{a^2} = 1$.

33. Show that the polar circle of the triangle formed by three tangents to an equilateral hyperbola touches the nine-point circle of the triangle formed by the points of contact at the centre of the curve.

34. Given a triangle circumscribed about a conic and the length of the axis major, show that the locus of the foci is a curve of the sixth order, of which the vertices of the triangles are nodes (see Ex. 29 (16)).

35. To find the locus of the centre of the circumscribing circle of a triangle circumscribed about a conic S, and inscribed in a conic S'.

It may be shown by the invariants that, if a triangle be circumscribed about a conic S, and inscribed in a conic S', it is also self-conjugate with regard to another fixed conic. Taking the values for S and S' in Salmon's *Conics*, Art. 376, we find the following value for the covariant F:

$$F \equiv -4\,(ghx^2 + hfy^2 + fgz^2) + 4\,(f+g+h)(fyz + gzx + hxy),$$

from which it is evident that $ghx^2 + hfy^2 + fgz^2 = 0$, being expressible in the form $2F - \Theta S' = 0$, represents a fixed conic.

Let us put

$$S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1, \quad S' = (a', b', c', f', g', h')\,(x, y, 1)^2;$$

then if we substitute the co-ordinates of the intersection of the tangents

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1 = 0, \quad \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta - 1 = 0, \quad (1)$$

in $S' = 0$, we get

$$\begin{aligned}(a'a^2 - b'b^2 + c')\cos\alpha\cos\beta + (b'b^2 - a'a^2 + c')\sin\alpha\sin\beta \\ + a'a^2 + b'b^2 + c' + 2h'ab\sin(\alpha+\beta) + 2g'a\,(\cos\alpha\ \cos\beta) \\ + 2f'b\,(\sin\alpha + \sin\beta) = 0.\end{aligned}$$

But this relation evidently expresses that the tangents (1) are conjugate with respect to the conic whose tangential equation is

$$a^2(a'a^2 - b'b^2 + c')\lambda^2 + b^2(b'b^2 - a'a^2 + c')\mu^2 + (a'a^2 + b'b^2 + c')\nu^2$$
$$+ 4h'a^2b^2\lambda\mu - 4g'a^2\nu\lambda - 4f'b^2\mu\nu \equiv \Sigma = 0\,; \qquad (2)$$

and triangles circumscribed about S and inscribed in S' are then evidently self-conjugate with regard to Σ. Now, since the circumscribed circle of the triangle cuts orthogonally the director circle of Σ, we have

$$(a'a^2 + b'b^2 + c')(x^2 + y^2 - r^2) + 4(g'a^2x + f'b^2y)$$
$$+ (a^2 + b^2)c' + (a^2 - b^2)(a'a^2 - b'b^2) = 0, \qquad (3)$$

where x, y are the co-ordinates of the centre of the circumscribing circle, and r is its radius (*Conics*, Art. 294, Ex.). We have also from Ex. 29 (15) $t^2t'^2 = 4b^2r^2$, or

$$(x^2 + y^2 + a^2 + b^2 - r^2)^2 = 4(a^2x^2 + b^2y^2 + a^2b^2)\,; \qquad (4)$$

hence, eliminating r between (3) and (4), we obtain

$$\{2g'a^2x + 2f'b^2y - a^2b^2(a' + b')\}^2$$
$$= (a'a^2 + b'b^2 + c')^2(a^2x^2 + b^2y^2 + a^2b^2). \quad (5)$$

If $a'a^2 + b'b^2 + c' = 0$, this conic becomes the square of a line, and in no other case will its discriminant vanish; for, if we form the discriminant of (5), we get, after dividing by $(a'a^2 + b'b^2 + c')^4$,

$$(a'a^2 + b'b^2 + c')^2 = 4a^2g'^2 + 4b^2f'^2 + a^2b^2(a' + b')^2,$$

and this combined with the invariant relation connecting S and S', viz.,

$$(a'a^2 + b'b^2 + c')^2 = 4a^2b^2(a'b' - h'^2) + 4a^2g'^2 + 4b^2f'^2,$$

gives $(a' - b')^2 + 4h'^2 = 0$, whence $a' = b'$, $h' = 0$, or S' must be a circle, in which case, of course, there is no locus. If $f' = 0$,

$$a'a^2 + b'b^2 + c' = \frac{\pm 2g'a^2}{\sqrt{(a^2 - b^2)}},$$

the locus becomes a circle.

36. To find the envelope of the circle circumscribing a triangle circumscribed about a conic S, and inscribed in a conic S'.

By the preceding example the circumscribing circle cuts orthogonally the circle (3), and has its centre on the conic (5); its envelope is, therefore, a *bicircular quartic:* a curve which is usually defined in this manner.

When the circle (3) and the conic (5) have double contact with each other, it can be shown that the quartic breaks up into two circles. If C is the circle (3) taken so that the coefficient of $x^2 + y^2$ is unity, two chords of intersection of (3) and (5) are easily seen to be $y^2 + (x \pm c)^2 - C = 0$, from which it is evident that (3) and (5) will have double contact, if the foci of S lie on C; in fact, putting

$$C \equiv x^2 + y^2 - 2\beta y - c^2,$$

the conic (5) becomes

$$(\beta^2 + c^2)y^2 - a^2C = 0,$$

and the envelope then breaks up into two circles passing through the foci. When C passes through the foci, we must have from (3),

$$g' = 0,\ (a^2 + b^2)c' + (a^2 - b^2)(a'a^2 - b'b^2) + (a^2 - b^2)(a'a^2 + b'b^2 + c') = 0,$$

or
$$c' + (a^2 - b^2)a' = 0,$$

from which it follows that S' must pass through the foci of S. Hence, if a triangle be circumscribed about a conic S, and inscribed in another conic passing through the foci of S, the circumscribing circle will touch two fixed circles also passing through the foci of S. If S becomes a circle, (3) and (5) have double contact with each other, and the envelope breaks up into two circles, which agrees with the result of Ex. 11.

37. To find the locus of the centre of a circle which touches the sides of a triangle circumscribed about a conic S, and inscribed in a conic S'.

Let $(a'', b'', c'', f'' g'', h'')(x, y, 1)^2 \equiv V = 0$ be the equation in x, y co-ordinates of the conic which we have called Σ in Ex. 35 (2); then, if x, y are the co-ordinates of the centre of the circle, and r its radius, we have

$$V = r^2 (a'' + b''),$$

since the circle touches the sides of a triangle self-conjugate with regard to V (*Conics*, Art. 375). But we have also the invariant relation connecting the circle with the conic S', viz.

$$r = \frac{ab}{c^2}\left\{\sqrt{\left(a^2 - \frac{c^2 x^2}{a^2}\right)} \pm \sqrt{\left(b^2 + \frac{c^2 y^2}{b^2}\right)}\right\},$$

Ex. 10 (3), if S' be written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0;$$

hence, eliminating r, the locus is a curve of the fourth order.

If the conic S' become a circle, the locus is a bicircular quartic; for if we put

$$x^2 + y^2 - 1 = S', \text{ and } V = (a, b, c, f, g, h)(x, y, 1)^2,$$

we have, from Euler's equation, $x^2 + y^2 - 1 = 2r$, which, combined with $V = r^2 (a + b)$, gives $(a + b)(x^2 + y^2 - 1)^2 = 4V$.

This quartic, when certain conditions are satisfied, can break up into two circles. Putting $h = 0$, and equating

$$(x^2 + y^2 - 1)^2 - \frac{4V}{a+b}$$

to the product of the circles $x^2 + y^2 + px + k$, $x^2 + y^2 - px + k'$, we get conditions which, combined with the invariant relation $c(a+b) = f^2 + g^2 + ab$ connecting V and S', give the further relations $f = 0$, $g^2 = a^2 - b^2$. The latter relations might, of course, be replaced by $g = 0$, $f^2 = b^2 - a^2$.

38. A triangle is circumscribed about a conic S and inscribed in a conic S', having double contact with the director circle of S; show that the polar circle of the triangle touches two fixed circles.

If S' is a parabola, show that the polar circles form a coaxial system.

39. To find the locus of the centre of the nine-point circle of a triangle circumscribed about a conic S and inscribed in a conic S'.

Let $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be the equation of S', and $(A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0$, the tangential equation of S; then, expressing that the chord $\frac{x}{a}\cos\frac{1}{2}(\alpha+\beta) + \frac{y}{b}\sin\frac{1}{2}(\alpha+\beta) - \cos\frac{1}{2}(\alpha-\beta) = 0$ of S' touches S, we get

$$\frac{A}{a^2}\cos^2\tfrac{1}{2}(\alpha+\beta) + \frac{B}{b^2}\sin^2\tfrac{1}{2}(\alpha+\beta) + C\cos^2\tfrac{1}{2}(\alpha-\beta)$$
$$-\frac{G}{a}(\cos\alpha + \cos\beta) - \frac{F}{b}(\sin\alpha + \sin\beta) + \frac{H}{ab}\sin(\alpha+\beta) = 0,$$

which, as at Ex. 29 (2), may be written in the form

$$L\cos\gamma + M\sin\gamma + N = 0, \qquad (1)$$

where

$$L = \frac{A}{a^2} + \frac{B}{b^2} - \frac{2G}{a} p - \frac{2F}{b} q + \tfrac{1}{2} C (p^2 + q^2 + 1), \qquad (2)$$

$$M = \left(\frac{A}{a^2} - \frac{B}{b^2}\right) \cos\phi + \frac{2H}{ab} \sin\phi + \frac{2G}{a} - Cp, \qquad (3)$$

$$N = \left(\frac{A}{a^2} - \frac{B}{b^2}\right) \sin\phi - \frac{2H}{ab} \cos\phi + \frac{2F}{b} - Cq. \qquad (4)$$

But, since (1) is true, when γ is replaced by α, β, respectively, we must have $L = M = N = 0$. Now, if x, y are the co-ordinates of the centre of the nine-point circle, we have, from Ex. 2, (1) and (2),

$$8ax = (3a^2 + b^2) p - c^2 \cos\phi, \; 8by = (a^2 + 3b^2) q - c^2 \sin\phi;$$

hence, from $M = N = 0$, we get expressions of the form

$$x = \lambda + \mu \cos\phi + \nu \sin\phi, \; y = \lambda' + \mu' \cos\phi + \nu' \sin\phi,$$

from which it follows that the locus is a conic.

If $\mu = \pm \nu'$, $\nu = \mp \mu'$, this conic will become a circle, and then we have $\quad H = 0, \; \dfrac{A}{a^2} - \dfrac{B}{b^2} = \dfrac{C(a \pm b)}{a \mp b}.$

If $\mu\nu' - \mu'\nu = 0$, the locus reduces to a line.

40. A triangle is circumscribed about a conic S and inscribed in a conic S'; show that the nine-point circle cuts orthogonally a fixed circle.

41. A triangle is circumscribed about a conic S and inscribed in a conic S'; show that the radical axis of the circumscribing and nine-point circles touches a fixed conic.

42. A triangle is circumscribed about the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and inscribed in the circle $(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0$; if s be half the sum of the angles subtended by the foci of the conic at the vertices of the triangle, to show that

$$\alpha^2 + \beta^2 - r^2 - c^2 = 2ar \cos s,\ c\beta = ar \sin s.$$

The relation at Ex. 29 (9) is, of course, also true for the imaginary foci, with the exception that b must be replaced by a; thus we have $a^2 t^2 = -2r\alpha\beta\gamma$. Now, if $x \cos \omega + y \sin \omega - p = 0$ be the equation of a tangent to the conic, we have $\alpha = p + c\sqrt{-1} \sin \omega$; but $p = a \cos \phi$, $c \sin \omega = a \sin \phi$, where ϕ is the angle the focal radius vector makes with the normal; hence, $\alpha = ae^{\phi\sqrt{-1}}$, and similar values for β, γ. Now it can be shown by geometrical considerations that the sum of the angles subtended by the foci at the points of contact is equal to the sum of the angles subtended at the vertices of the triangle; hence, we have $t^2 = 2are^{s\sqrt{-1}}$; therefore, putting $t^2 = \alpha^2 + (\beta + c\sqrt{-1})^2$, and equating real and imaginary parts, we obtain the results required.

43. To find the locus of the centroid of an equilateral triangle circumscribed about a conic.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ be the equation of the conic; then, from the invariant relation connecting the conic with the circumscribing circle of the triangle, we have, from Ex. 29 (15),

$$R^4 - 2R^2(x^2 + y^2 + a^2 + b^2) + (x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 = 0, \quad (1)$$

where R is the radius of the circle, and x, y the co-ordinates of its centre. Again, from the invariant relation connecting the conic with the polar circle, we have

$$x^2 + y^2 = a^2 + b^2 + M, \qquad (2)$$

where M is the rectangle under the segments of the perpendiculars. But, for an equilateral triangle, $M = -\frac{1}{2}R^2$; hence, eliminating R between the equations (1), (2), we get

$$(3x^2 + 3y^2 - a^2 - b^2)^2 = 4(a^2x^2 + b^2y^2 + a^2b^2).$$

44. To find the equations of the circles touching the sides of a triangle circumscribed about a conic.

Let the equation of the conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \equiv S = 0$, and that of one of the circles $(x - x')^2 + (y - y')^2 - r^2 \equiv S' = 0$; then the conditions that the line $\lambda x + \mu y - 1 = 0$ should touch S and S' are

$$a^2\lambda^2 + b^2\mu^2 - 1 \equiv \Sigma = 0, \ (x'\lambda + y'\mu - 1)^2 - r^2(\lambda^2 + \mu^2) \equiv \Sigma' = 0,$$

respectively.

Now, if we form the discriminant of $\Sigma + \frac{h^2}{r^2}\Sigma'$, we get

$$\frac{x'^2}{a^2 - h^2} + \frac{y'^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0, \qquad (1)$$

which is identical with the equation already obtained in Ex. 1 for the discriminant of $S' - h^2 S$. But, writing $\Sigma + \frac{h^2}{r^2}\Sigma'$ in the form $(a^2 - h^2)\lambda^2 + (b^2 - h^2)\mu^2 - 1 + \frac{h^2}{r^2}(x'\lambda + y'\mu - 1)^2$, we see that when $\Sigma + \frac{h^2}{r^2}\Sigma'$ represents two points, these points must lie on the confocal conic $(a^2 - h^2)\lambda^2 + (b^2 - h^2)\mu^2 - 1 = 0$,

or $$\frac{x^2}{a^2 - h^2} + \frac{y^2}{b^2 - h^2}\ 1 = 0. \qquad (2)$$

Hence it follows at once that the extremities of a diagonal of the quadrilateral formed by the common tangents of S and S' lie on the same confocal conic, three conics such as (2) corresponding to the roots of the equation (1). From (2), if ν be half the major axis of a confocal conic passing through a vertex of the triangle, we have $h^2 = a^2 - \nu^2$; and the equation (1) then becomes

$$\frac{x'^2}{\nu^2} + \frac{y'^2}{c^2 - \nu^2} + \frac{r^2}{a^2 - \nu^2} - 1 = 0. \qquad (3)$$

This equation (3) gives the semiaxes ν_1, ν_2, ν_3 of three confocal conics passing through the vertices of the triangle; and from its absolute term we get

$$acx' = \nu_1\nu_2\nu_3, \qquad (4)$$

and similarly from the absolute term of the equation whose roots are

$$c^2 - \nu_1^2,\ c^2 - \nu_2^2,\ c^2 - \nu_3^2,\ bcy' = \sqrt{(c^2-\nu_1^2)(c^2-\nu_2^2)(c^2-\nu_3^2)}. \quad (5)$$

Also from the coefficient of ν^4 we have

$$x'^2 + y'^2 - r^2 = \nu_1^2 + \nu_2^2 + \nu_3^2 - a^2 - c^2;$$

hence the equation

$$x^2 + y^2 - \frac{2\nu_1\nu_2\nu_3 x}{ac} - \frac{2}{bc}\sqrt{(c^2-\nu_1^2)(c^2-\nu_2^2)(c^2-\nu_3^2)}\,y + \nu_1^2 + \nu_2^2 + \nu_3^2 - a^2 - c^2 = 0 \qquad (6)$$

represents one of the circles touching the sides of the triangle. If S is an ellipse, these three confocals are hyperbolæ for the circle inscribed in the triangle. If μ_1, μ_2, μ_3 are the semiaxes of the three confocal ellipses passing through the vertices of the triangle, the equation

$$x^2 + y^2 - \frac{2\mu_1\mu_2\nu_3 x}{ac} - \frac{2}{bc}\sqrt{(\mu_1^2-c^2)(\mu_2^2-c^2)(c^2-\nu_3^2)}\,y + \mu_1^2 + \mu_2^2 + \nu_3^2 - a^2 - c^2 = 0 \qquad (7)$$

represents one of the exscribed circles.

From the absolute term of the equation (1) in h^2 we get for the radius r of the inscribed circle

$$r = \frac{\sqrt{(a^2-\nu_1^2)(a^2-\nu_2^2)(a^2-\nu_3^2)}}{ab}, \qquad (8)$$

and for the radius r' of an exscribed circle,

$$r' = \frac{\sqrt{(\mu_1^2 - a^2)(\mu_2^2 - a^2)(a^2 - \nu_3^2)}}{ab}, \qquad (9)$$

From these expressions for the radii we deduce, if s be the semi-perimeter of the triangle,

$$s = \frac{\sqrt{(\mu_1^2 - a^2)(\mu_2^2 - a^2)(\mu_3^2 - a^2)}}{ab}. \qquad (10)$$

By means of elliptic functions the equation of the circle inscribed in a circumscribed triangle can be written in a form similar to that of the circumscribing circle of an inscribed triangle. Putting $x' = a \sin\phi$, $y' = b \cos\phi$ for a point on S, the equation of the tangent at x', y' is

$$\frac{x}{a}\sin\phi + \frac{y}{b}\cos\phi - 1 = 0\ ;$$

and if we suppose this tangent to pass through the point

$$x = \mu \sin\psi, \quad y = \sqrt{(\mu^2 - c^2)}\ \cos\psi,$$

on the confocal ellipse

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - c^2} - 1 = 0,$$

we have

$$\frac{\mu}{a}\sin\phi\ \sin\psi + \frac{\sqrt{(\mu^2 - c^2)}}{b}\cos\phi\ \cos\psi = 1. \qquad (11)$$

Comparing this equation (11) with

$$\cos\phi\ \cos\psi + \sin\phi\ \sin\psi\sqrt{(1 - k^2\sin^2\sigma)} = \cos\sigma,$$

we get
$$k = e, \quad \sin^2\sigma = \frac{\mu^2 - a^2}{\mu^2 - c^2}\ ; \qquad (12)$$

hence we have $F(\phi) - F(\psi) = F(\sigma)$, and for the other tangent $\frac{x}{a}\cos\phi' + \frac{y}{b}\sin\phi' - 1 = 0$ through xy $F(\psi) - F(\phi') = F(\sigma)$, whence

$$2F(\psi) = F(\phi) + F(\phi'). \tag{13}$$

But if ν be the semiaxis of the confocal hyperbola passing through xy, we have $x = \frac{\mu\nu}{c} = \mu \sin\psi$, whence $\nu = c\sin\psi = c \operatorname{sn}\frac{1}{2}(u_1 + u_2)$ from (13), if we write $F(\phi) = u_1$, $F(\phi') = u_2$. Thus we have with the usual notation of elliptic functions

$$x' = \frac{c^2}{a} \operatorname{sn}\tfrac{1}{2}(u_1 + u_2) \operatorname{sn}\tfrac{1}{2}(u_2 + u_3) \operatorname{sn}\tfrac{1}{2}(u_3 + u_1),$$

$$y' = -\frac{c^2}{b} \operatorname{cn}\tfrac{1}{2}(u_1 + u_2) \operatorname{cn}\tfrac{1}{2}(u_2 + u_3) \operatorname{cn}\tfrac{1}{2}(u_3 + u_1),$$

$$r = \frac{a^2}{b} \operatorname{dn}\tfrac{1}{2}(u_1 + u_2) \operatorname{dn}\tfrac{1}{2}(u_2 + u_3) \operatorname{dn}\tfrac{1}{2}(u_3 + u_1).$$

From (12) we have

$$\mu = \frac{a\sqrt{(1 - e^2 \sin^2\sigma)}}{\cos\sigma} = \frac{a \operatorname{dn}\frac{1}{2}(u_1 - u_2)}{\operatorname{cn}\frac{1}{2}(u_1 - u_2)},$$

and

$$\sqrt{(\mu^2 - c^2)} = \frac{b}{\operatorname{cn}\frac{1}{2}(u_1 - u_2)};$$

hence, for an exscribed circle,

$$x'' = a\,\frac{\operatorname{dn}\frac{1}{2}(u_1 - u_2) \operatorname{dn}\frac{1}{2}(u_2 - u_3) \operatorname{sn}\frac{1}{2}(u_3 + u_1)}{\operatorname{cn}\frac{1}{2}(u_1 - u_2) \operatorname{cn}\frac{1}{2}(u_2 - u_3)},$$

$$y'' = b\,\frac{\operatorname{sn}\frac{1}{2}(u_3 + u_1)}{\operatorname{cn}\frac{1}{2}(u_1 - u_2) \operatorname{cn}\frac{1}{2}(u_2 - u_3)},$$

$$r' = b\,\frac{\operatorname{sn}\frac{1}{2}(u_1 - u_2) \operatorname{sn}\frac{1}{2}(u_2 - u_3) \operatorname{dn}\frac{1}{2}(u_3 + u_1)}{\operatorname{cn}\frac{1}{2}(u_1 - u_2) \operatorname{cn}\frac{1}{2}(u_2 - u_3)},$$

and similar expressions for the other exscribed circles.

45. Show, by comparing the equations of the circles in Ex. 1 and Ex. 44, that the points of contact of tangents to a conic S, which are parallel to its chords of intersection with a circle S', lie on confocal conics passing through the points of intersection of the common tangents of S and S'.

46. If two vertices of a triangle circumscribed about an ellipse move along confocal hyperbolæ, show that the locus of the centre of the inscribed circle is a concentric ellipse.

If two vertices move along confocal ellipses, show that the centres of each of the inscribed circles lie on concentric ellipses.

47. A circle touches the tangents drawn from a variable point on a conic S to a confocal conic, and cuts orthogonally a circle S'; show that its centre lies on a conic passing through the intersection of S and S'.

48. Given a triangle circumscribed about a conic and a point on the curve, to find the envelope of the director circle.

If the conic referred to the triangle be written in the form $\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{m\gamma} = 0$, the equation of the director circle is $lbcS_1 + mcaS_2 + nabS_3 = 0$ (1) (*Conics*, Art. 383), where S_1, S_2, S_3 are the three circles described on the sides of the triangle as diameters. But the envelope of the circle (1), subject to the condition $\sqrt{l\alpha'} + \sqrt{m\beta'} + \sqrt{n\gamma'} = 0$, is $a\alpha' S_2 S_3 + b\beta' S_3 S_1 + c\gamma' S_1 S_2 = 0$ (2), which represents a bicircular quartic passing through the vertices and the feet of the perpendiculars. We can also see that the envelope is a bicircular quartic, from the fact that it cuts orthogonally the polar circle, and has its centre on the fixed conic (*Conics*, Art. 293, Ex. 2)

$$\sqrt{\{a\alpha'(b\beta + c\gamma - a\alpha)\}} + \sqrt{\{b\beta'(c\gamma + a\alpha - b\beta)\}} + \sqrt{\{c\gamma'(a\alpha + b\beta - c\gamma)\}} = 0. \qquad (3)$$

When the quartic breaks up into two circles we see from (2) that these circles must be the circumscribing and nine-point circles of the triangle. If the fixed point is at infinity, the quartic can break up into one of the sides of the triangle, and the circle described on the opposite vertex and the intersection of the perpendiculars as diameter.

To find the point through which the curve passes in the first of these cases, we express that the radical axis of the director circle and the circumscribing circle (*Conics*, Art. 383)

$$l\alpha \cot A + m\beta \cot B + n\gamma \cot C = 0$$

touches the circumscribing circle, when we obtain

$$\sqrt{l \cos A} + \sqrt{m \cos B} + \sqrt{n \cos C} = 0,$$

which is evidently the condition that the conic should pass through the centre of the circumscribing circle; hence, if a triangle be circumscribed about a conic, so as to have the centre of the circumscribing circle on the curve, the circumscribing and nine-point circles will both touch the director circle. We can verify this result by means of the invariants; for if we write the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and the circumscribing circle $(x - x')^2 + (y - y')^2 - r^2 \equiv S = 0$, we have

$$(x'^2 + y'^2 + a^2 + b^2 - r^2)^2 = 4\,(a^2x'^2 + b^2y'^2 + a^2b^2),$$

which, combined with $r + \sqrt{(a^2 + b^2)} = \sqrt{(x'^2 + y'^2)}$, the condition that the director circle should touch S, gives

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0\text{; therefore, \&c.}$$

If a tangent to the curve touch the director circle, we can easily see that it must be perpendicular to an asymptote; hence, if a triangle be circumscribed about an hyperbola, so that one of the sides is perpendicular to an asymptote, the director circle will touch that side, and also from (2) the circle described on the opposite vertex and the intersection of perpendiculars as diameter.

In all the cases in which the envelope breaks up into curves of lower degree, the conic (3) has double contact with the polar circle; in fact, putting $\alpha' = \cos A$, $\beta' = \cos B$, $\gamma' = \cos C$ in (3), it may be written in the form

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$$
$$- 2 \sin A \sin B \sin C (\alpha \cos A + \beta \cos B + \gamma \cos C)^2 = 0;$$

and putting $\alpha' = \cos B$, $\beta' = \cos A$, $\gamma' = -1$ for the point at infinity on the perpendicular to γ, (3) becomes

$$\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C - 2 \sin A \sin B \sin C\gamma^2 = 0.$$

49. Given a triangle circumscribed about a conic, and that a directrix passes through a fixed point, to find the locus of the corresponding focus.

If ρ_1 be the distance of the focus from a vertex A of the triangle, ϕ_1 the angle subtended at the focus by the side opposite A, α the perpendicular from A on the directrix, and e the eccentricity of the curve, we have (*Conics*, Art. 191)

$$e\alpha = \rho_1 \cos \phi_1 = \frac{\rho_1 S_1}{\rho_2 \rho_3}, \qquad (1)$$

where S_1 is the square of the tangent drawn from the focus to the circle described on the side opposite A as diameter, and ρ_2, ρ_3 are the distances of the focus from the other vertices. Now, if α, β, γ are the perpendiculars from the vertices on the directrix, we must have $l\alpha + m\beta + n\gamma = 0$, where l, m, n are the areal co-ordinates of the fixed point; hence the focus lies on

$$l\rho_1^2 S_1 + m\rho_2^2 S_2 + n\rho_3^2 S_3 = 0, \qquad (2)$$

which represents a bicircular quartic circumscribing the triangle.

If the fixed point is the intersection of the perpendiculars, the quartic (2) must be divisible by the circumscribing circle; for a parabola always satisfies this condition, and then the focus lies on the circumscribing circle. The other factor is the polar circle of the triangle; for this circle is orthogonal to the director circle, and it can easily be proved that a circle orthogonal to the director circle, and having its centre on a directrix, passes through the corresponding focus.

If the directrix is parallel to a given line, the locus is a circular cubic, as may be also proved as follows:—Let x, y, z be the perpendiculars from the focus on the sides of the triangle; α, β, γ the angles between the sides and the given line; then we have

$$x^2 + 2cx \cos\alpha - b^2 = 0,\ y^2 + 2cy \cos\beta - b^2 = 0,\ z^2 + 2cz \cos\gamma - b^2 = 0;$$

and, eliminating b and c from these equations, we get

$$x \cos\alpha\,(y^2 - z^2) + y \cos\beta\,(z^2 - x^2) + z \cos\gamma\,(x^2 - y^2) = 0,$$

which represents a circular cubic passing through the vertices and the centres of the circles touching the sides.

50. A triangle is circumscribed about a conic so that the lines, which are drawn from each vertex to the point of the opposite side with the circle escribed to that side, intersect on the curve: show that the circle inscribed in the triangle formed by the middle points of the sides passes through the centre of the conic.

51. A conic whose foci are F, F' is inscribed in a triangle; if F lie on the polar circle of the triangle, show that an equilateral hyperbola can be described, having F' for centre, and passing through the feet of the perpendiculars from F' on the sides.

If F lie on one of the circles touching the sides, show that a parabola can be described having F' for focus, and passing through the feet of the perpendiculars from F' on the sides.

52. Show that the intersection of the perpendiculars of a triangle formed by three tangents to an equilateral hyperbola and the centre of the circle passing through the points of contact of the tangents are conjugate with respect to the curve.

53. A triangle is circumscribed about the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, so that the radical axis of the circumscribing and polar circles passes through the centre of the curve; show that the centre of the circumscribing circle lies on the conic

$$a^2x^2 + b^2y^2 = a^4 + b^4 + a^2b^2.$$

III.—SELF-CONJUGATE TRIANGLES.

54. To find the equation of the polar circle of a triangle self-conjugate with regard to a conic.

We have seen in Ex. 1 that when a conic and circle are written in the forms $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$,

$$S' \equiv (x - x')^2 + (y - y')^2 - r^2 = 0,$$

the discriminant of $S - h^2 S'$ is given by the equation

$$\frac{x'^2}{a^2 - h^2} + \frac{y'^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0. \qquad (1)$$

But if S and S' are referred to their common self-conjugate triangle, we have

$$S \equiv -\frac{1}{2\Delta}\left(\frac{a\alpha^2}{\alpha'} + \frac{b\beta^2}{\beta'} + \frac{c\gamma^2}{\gamma'}\right), \qquad (2)$$

$$S' \equiv \frac{\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C}{2 \sin A \sin B \sin C}, \qquad (3)$$

where α', β', γ' are the co-ordinates of the centre of S, S being taken so that the result of substituting these values is negative unity, and S' so that the coefficient of $x^2 + y^2$ is unity.

But when S and S' are written in the forms (2) and (3), the discriminant of $S - h^2 S'$ is

$$(h^2 + 2R\alpha' \cos A)\,(h^2 + 2R\beta' \cos B)\,(h^2 + 2R\gamma' \cos C),$$

where R is the radius of the circumscribing circle. This equation must coincide with (1); hence, if h_1^2, h_2^2, h_3^2 are the roots of (1), we have

$$h_1^2 h_2^2 h_3^2 = -8R^3 \alpha' \beta' \gamma' \cos A \cos B \cos C;$$

but $h_1 h_2 h_3 = abr$, as we have seen in Ex. 1, and

$$r^2 = -4R^2 \cos A \cos B \cos C;$$

therefore $$2R = \frac{a^2 b^2}{\alpha' \beta' \gamma'}. \qquad (4)$$

Again, if x_1, y_1, &c., are the co-ordinates of the vertices of the triangle, we have $\cos A = -\beta'\gamma'\left(\frac{x_2 x_3}{a^4} + \frac{y_2 y_3}{b^4}\right)$; therefore

$$h_1^2 = -2R\alpha' \cos A = 2R\alpha'\beta'\gamma'\left(\frac{x_2 x_3}{a^4} + \frac{y_2 y_3}{b^4}\right) = a^2 + b^2 - x_2 x_3 - y_2 y_3$$

from (4), since

$$\frac{x_2 x_3}{a^2} + \frac{y_2 y_3}{b^2} = 1, \text{ and } a^2 - h_1^2 = e^2 x_2 x_3,\ h_1^2 - b^2 = \frac{c^2}{b^2} y_2 y_3;$$

hence, since as in Ex. 1,

$$x' = \frac{1}{ac}\sqrt{\{(a^2 - h_1^2)(a^2 - h_2^2)(a^2 - h_3^2)\}},$$

we obtain $$x' = \frac{c^2}{a^4} x_1 x_2 x_3,$$

and, similarly, $$y' = -\frac{c^2}{b^4} y_1 y_2 y_3;$$

also $$x'^2 + y'^2 - r' = a^2 + b^2 - (h_1^2 + h_2^2 + h_3^2)$$
$$= e^2(x_1 x_2 + x_2 x_3 + x_3 x_1) - (a^2 + c^2).$$

Thus, finally, the equation of S' can be written

$$x^2 + y^2 - \frac{2c^2}{a^4} x_1 x_2 x_3 x + \frac{2c^2}{b^4} y_1 y_2 y_3 y + e^2(x_1 x_2 + x_2 x_3 + x_3 x_1) - (a^2 + c^2) = 0.$$

55. To find the equation of the circle circumscribing the common self-conjugate triangle of two conics.

Let S and S' be the two conics, then the director circles of all the conics of the system $S + kS' = 0$ cut orthogonally the circle circumscribing their common self-conjugate triangle. But the director circle of $S + kS'$ is (*Conics*, Art. 294)

$$C\lambda^2 + \Sigma\lambda + C' = 0,$$

where C and C' are the director circles of S and S', and

$$\begin{aligned}\Sigma \equiv (a'b + b'a - 2hh')(x^2 + y^2) - 2x(fh' + f'h - bg' - b'g) \\ - 2y(gh' + g'h - af' - a'f) \\ + c'(a + b) + c(a' + b') - 2gg' - 2ff'\end{aligned}$$

is the director circle of the covariant conic Φ; hence we obtain the equation of the circle required by forming that of the circle cutting C, C' and Σ at right angles (*Conics*, Art. 132, (*a*)).

56. To find the locus of the centroid of an equilateral triangle self-conjugate with regard to a conic.

Let R be the radius of the circumscribing circle, and r that of the inscribed circle, then if the equation of the conic is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have, from the invariants,

$$x^2 + y^2 = a^2 + b^2 + R^2,\ b^2x^2 + a^2y^2 - a^2b^2 = (a^2 + b^2)r^2 = \tfrac{1}{4}(a^2 + b^2)R^2;$$

hence the equation of the locus is

$$(a^2 - 3b^2)x^2 - (3a^2 - b^2)y^2 = (a^2 - b^2)^2.$$

57. Show that the circle circumscribing an equilateral triangle self-conjugate with regard to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

has double contact with the bicircular quartic

$$(x^2 + y^2 + a^2 + b^2)^2 = 4(a^2 - b^2)^2 \left\{ \frac{x^2}{a^2 - 3b^2} - \frac{y^2}{3a^2 - b^2} \right\}.$$

58. To find the locus of the vertices of equilateral triangles self-conjugate with regard to a conic.

Let the conic be $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and let $x \cos \alpha + y \sin \alpha = 0$, $x \cos \beta + y \sin \beta = 0$ be lines through the origin parallel to the two sides of the triangle which meet in the vertex x, y. Then, expressing that these sides form a harmonic pencil with the tangents to the curve from x, y, we get

$$\cos \alpha \cos \beta (x^2 - a^2) + \sin \alpha \sin \beta (y^2 - b^2) - xy \sin (\alpha + \beta) = 0,$$

or $$(x^2 + y^2 - a^2 - b^2) \cos (\alpha - \beta) - (x^2 - y^2 - c^2) \cos (\alpha + \beta) - 2xy \sin (\alpha + \beta) = 0. \quad (1)$$

But $$\cos (\alpha - \beta) = \cos \frac{\pi}{3} = \tfrac{1}{2}, \text{ and } \alpha + \beta = 2\omega,$$

where $$\cos \omega = \frac{\frac{x}{a^2}}{\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}}, \quad \sin \omega = \frac{\frac{y}{b^2}}{\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)}};$$

hence (1) becomes

$$\tfrac{1}{2}(x^2 + y^2 - a^2 - b)\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = (x^2 - y^2 - c^2)\left(\frac{x^2}{a^4} - \frac{y^2}{b^4}\right) + \frac{4x^2y^2}{a^2b^2},$$

or $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 - \frac{3c^4x^2y^2}{a^4b^4} + (3b^2 - a^2)\frac{x^2}{a^4} + (3a^2 - b^2)\frac{y^2}{b^4} = 0,$$

which represents a quartic curve with a node at the origin.

If $a^2 - 3b^2 = 0$, the locus breaks up into two imaginary conics.

For the parabola $y^2 - 4mx = 0$ the locus is

$$y^2(3x + 7m) - 4m^2(x - 3m) = 0.$$

59. Given a triangle self-conjugate with regard to a conic, if a directrix passes through a fixed point, to find the locus of the corresponding focus.

If a conic be referred to two lines x, y, at right angles to each other through a focus, and γ the corresponding directrix, the equation of the curve is $x^2 + y^2 = e^2\gamma^2$; hence, if two points are conjugate, we have $x_1x_2 + y_1y_2 = e^2\gamma_1\gamma_2$, which may be written $S_3 = e^2\beta\gamma$, where β, γ are the perpendiculars from the points on the directrix, and S_3 is the square of the tangent drawn from the focus to the circle described on the line joining the points as diameter. If, then, we are given a self-conjugate triangle, and a directrix pass through the fixed point determined by the equation $l\alpha + m\beta + n\gamma = 0$, the corresponding focus will lie on the bicircular quartic

$$lS_2S_3 + mS_3S_1 + nS_1S_2 = 0.$$

This quartic coincides with the envelope of the director circle in Ex. 48, and will break up into circles in the same cases.

60. Given a self-conjugate triangle with regard to a conic: if a directrix touch a conic U inscribed in the triangle, show that the corresponding focus lies on the director circle of U.

61. A triangle is self-conjugate with regard to a conic; show that the feet of its perpendiculars form a triangle circumscribed about a confocal conic.

62. Given a self-conjugate triangle with regard to a conic, and that half the length of the axis major is equal to the radius of the polar circle, to find the envelope of the axis minor.

If two points are conjugate with respect to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have $$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} - 1 = 0,$$

or $$x_1 x_2 + y_1 y_2 = b^2 + e^2 x_1 x_2,$$

which may be written $S_3 = b^2 + e^2 \beta\gamma,$ (1)

where S_3 is the square of the tangent drawn from the centre to the circle described on the line joining the points as diameter, and β, γ are the perpendiculars from the points on the axis minor. Now, if A, B, C are the angles and Δ the area of the triangle, we can verify the following identical relation:

$$S_1 \tan A + S_2 \tan B + S_3 \tan C = 2\Delta + t^2 \tan A \tan B \tan C, \quad (2)$$

where t is the tangent drawn from a point to the circumscribing circle of the triangle. But $t^2 = a^2 + b^2$, since the circumscribing circle cuts the director circle orthogonally; and $r^2 = -2\Delta \cot A \cot B \cot C$, where r is the radius of the polar circle; hence, from (1), and similar relations for the other sides, (2) gives

$$\beta\gamma \tan A + \gamma\alpha \tan B + \alpha\beta \tan C = \frac{2\Delta}{e^2}\left(1 - \frac{a^2}{r^2}\right);$$

and the envelope, putting $r = a$, is

$$\beta\gamma \tan A + \gamma\alpha \tan B + \alpha\beta \tan C = 0, \quad (3)$$

which represents a conic inscribed in the triangle and concentric with the circumscribing circle.

63. Given a self-conjugate triangle with regard to an ellipse, and that the latus-rectum is equal to the radius of the polar circle, show that the major axis touches a conic confocal with the conic (3) in the preceding example.

64. A circle S touches the sides of a triangle self-conjugate with regard to a conic U; show that the centre of S lies on the equilateral hyperbola having double contact with U at a pair of points which lie on a tangent to S.

65. A circle S touches the sides of a triangle self-conjugate with regard to a conic $U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; if the tangents drawn from the centre of S to U touch at the points P, Q, show that the tangents from P, Q to the confocal conic $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$ all touch S.

66. If circles touching the sides of triangles self-conjugate with regard to the conic $U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ have their centres on a fixed line, show that they have double contact with a fixed conic whose foci are the points where the fixed line meets U.

And, conversely, if a circle have double contact with U, the chord of contact being perpendicular to the transverse axis, show that it is inscribed in triangles self-conjugate with regard to the conic

$$b^2 (x^2 - c^2) - a^2 y^2 + 2fy = 0,$$

where f is an arbitrary constant.

67. A circle is inscribed in a triangle self-conjugate with regard to the conic $U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and has its centre on

the director circle of U; show that it touches the conic

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2+b^2}.$$

68. If two circles touch the sides of triangles self-conjugate with regard to a conic U, show that their centres of similitude are conjugate with respect to U.

69. A circle inscribed in a triangle, self-conjugate with regard to a hyperbola, cuts the hyperbola orthogonally at a point P; show that P must be the point of contact of a tangent perpendicular to an asymptote.

IV.—TRIANGLES FORMED BY TWO TANGENTS AND THEIR CHORD OF CONTACT.

70. To find the equation of the circle circumscribing the triangle formed by the tangents drawn from the point x', y to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, and their chord of contact.

The equation of the circle must be of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + (lx + my + n)\left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right) = 0. \quad (1)$$

Substituting in this equation the co-ordinates of the point x', y', we get $n = -(lx' + my' + 1)$, and the conditions that (1) should represent a circle give

$$\frac{ly'}{b^2} + \frac{mx'}{a^2} = 0, \quad \frac{1 + lx'}{a^2} = \frac{1 + my'}{b^2}.$$

Hence the equation (1) becomes

$$a^2 b^2\left(\frac{x'^2}{a^4} + \frac{y'^2}{b^4}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + c^2\left\{\frac{xx'}{a^2} - \frac{yy'}{b^2} - \frac{(x'^2 + y'^2)}{c^2}\right\}$$
$$+ \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1\right) = 0,$$

or $$\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)(x^2 + y^2) - \frac{xx'}{a^2}(x'^2 + y'^2 + c^2) - \frac{yy'}{b^2}(x'^2 + y'^2 - c^2)$$
$$+ c^2\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right) = 0. \quad (2)$$

Also, if R be the radius of the circle, we have

$$4R^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2 = \frac{x'^2}{a^4}(x'^2+y'^2+c^2)^2 + \frac{y'^2}{b^4}(x'^2+y'^2-c^2)^2$$

$$-4c^2\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)$$

$$=\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)\{(x^2+y^2)^2-2c^2(x^2-y^2)+c^4\};$$

hence
$$2R = \frac{\rho\rho'}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}}\sqrt{\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)}, \tag{3}$$

where ρ, ρ' are the distances of x', y' from the foci. If θ is the angle under which this circle cuts the director circle, we have

$$c^2\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right) = \left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)\{a^2+b^2-2R\cos\theta\sqrt{(a^2+b^2)}\};$$

therefore, from (3),

$$(a^2+b^2)\rho^2\rho'^2\cos^2\theta = 4(b^4x'^2+a^4y'^2),$$

or
$$\rho^2\rho'^2\sin^2\theta = (x'^2+y'^2-a^2-b^2)\left\{x'^2+y'^2-\frac{(a^2-b^2)^2}{a^2+b^2}\right\}$$

Hence, if x', y' lie on the director circle or the inverse of the director circle with respect to the circle described on the line joining the foci as diameter, the variable circle touches the director circle.

71. Since the equation (2) in the preceding example is unaltered, if we substitute $\frac{c^2x'}{x'^2+y'^2}$, $\frac{-c^2y'}{x'^2+y'^2}$ for x', y', respectively, it follows that two points connected by the reciprocal relations $x_1y_2+y_1x_2=0$, $x_1x_2-y_1y_2-c^2=0$, and the points of contact of tangents from them to the curve lie on the same circle. If we suppose one of these points to lie on the curve, we see that the circle passing through any point P on the quartic

$$(x^2 + y^2)^2 - c^4\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0,$$

and the points of contact of the tangents from P, touches the curve.

72. From the equation (2) we can readily find the invariant relation connecting the circle with the curve. If we make the equation (2) identical with $x^2+y^2-2\alpha x-2\beta y+k^2=0$,

we have
$$\frac{4a^2\alpha^2}{k^2+c^2} = \frac{(x'^2+y'^2+c^2)^2}{2c^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)},$$

$$\frac{4b^2\beta^2}{k^2-c^2} = -\frac{(x'^2+y'^2-c^2)^2}{2c^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)},$$

$$a^2 + b^2 + k^2 = \frac{2(x'^2+y'^2)}{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}};$$

hence
$$\frac{a^2\alpha^2}{k^2+c^2} + \frac{b^2\beta^2}{k^2-c^2} = \tfrac{1}{4}(a^2+b^2+k^2).$$

This is the same equation as that which would be obtained by expressing that the roots of the equation (10) in Ex. 1 are connected by the relation $h_1^2 + h_2^2 = h_3^2$.

73. The circle passing through a point P, and the points of contact of the tangents from P to a conic, cuts orthogonally a fixed circle J; show that the locus of P is a circular cubic, and that of the centre of the circle a cubic.

If J passes through the foci, show that P lies on a circle passing through the foci, and the centre of the circle on a fixed conic, both of the loci, in this case, being divisible by the axis major.

74. The circle passing through a point P and the points of contact of the tangents from P to the conic

$$V \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

has its centre on the director circle of V; show that the locus of P is the bicircular quartic

$$(x^2 + y^2)^2 - 2x^2 (3a^2 - b^2) - 2y^2 (3b^2 - a^2) + (a^2 - b^2)^2 = 0.$$

75. To find the co-ordinates of the centroid of the triangle formed by the tangents from x', y' to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

and their chord of contact.

Let α, β be the co-ordinates of the middle point of the chord of contact, then we have

$$3x = 2\alpha + x', \;\; 3y = 2\beta + y';$$

but it can be easily seen that

$$\alpha = \frac{x'}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}, \; \beta = \frac{y'}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}};$$

hence we have

$$x = \tfrac{1}{3} x' \left(1 + \frac{2}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}\right),$$

$$y = \tfrac{1}{3} y' \left(1 + \frac{2}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}\right). \qquad (1)$$

If the centroid of the triangle lies on the curve, we find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 4 = 0$$

for the locus of x', y', after dividing by the equation of the curve.

76. To find the co-ordinates of the intersection of the perpendiculars of the triangle formed by the tangents from x', y' to the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ and their chord of contact.

Let x_1, y_1 be the co-ordinates of the centroid, and x_2, y_2 those of the centre of the circumscribing circle, then we have, by a known geometrical relation,

$$x = 3x_2 - 2x_1, \quad y = 3y_2 - 2y_1,$$

where x, y are the co-ordinates of the point we seek.

But x_2, y_2 are expressed in terms of x', y' in Ex. 70 (2), and x_1, y_1 in Ex. 75 (1); hence we obtain

$$x = \frac{\frac{x'}{a^2}\left(a^2 + b^2 + \frac{c^2 y'^2}{b^2}\right)}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}, \qquad (1)$$

$$y = \frac{\frac{y'}{b^2}\left(a^2 + b^2 - \frac{c^2 x'^2}{a^2}\right)}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}.$$

Since from the equations (1) we have $xx' + yy' = a^2 + b^2$, it follows that these two points are conjugate with respect to the director circle.

We have also $$\frac{a^2 x}{x'} - \frac{b^2 y}{y'} - c^2 = 0;$$

hence if we are given x, y, we determine x', y' as the intersection of the polar of x, y with regard to the director circle and the equilateral hyperbola which passes through the feet of the normals to the curve from x, y (*Conics*, Art. 181, Ex. 1).

77. Tangents are drawn from a point of the curve

$$(a^2+b^2)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)-(a^2-b^2)^2\frac{x^2y^2}{a^4b^4}=0$$

to the conic $$U\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0;$$

show that they form with their cord of contact a triangle whose intersection of perpendiculars lies on the director circle of U.

78. To find the equation of the polar circle of the triangle formed by two tangents to a conic and their chord of contact.

The co-ordinates of the centre are given at Ex. 76 (1), and we find the absolute term by expressing that x', y' is the pole of $\frac{xx'}{a^2}+\frac{yy'}{b^2}-1=0$ with regard to the circle; hence we find for the equation sought—

$$\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)(x^2+y^2)-\frac{2x'}{a^2}\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right)x-\frac{2y'}{b^2}\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right)y$$
$$+a^2+b^2+\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2=0. \quad (1)$$

If r is the radius of the circle, we find then

$$r^2=\frac{a^2b^2(x'^2+y'^2-a^2-b^2)\left(\frac{x'^2}{a^6}+\frac{y'^2}{b^6}-\frac{c^4x'^2y'^2}{a^6b^6}\right)}{\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)^2}. \quad (2)$$

The equation (1) can also be written in the form

$$\frac{1}{a^2}\left(a^2 + b^2 + \frac{c^2 y'^2}{b^2}\right)(x - x')^2 + \frac{1}{b^2}\left(a^2 + b^2 - \frac{c^2 x'^2}{a^2}\right) y - y')^2$$
$$+ (x'^2 + y'^2 - a^2 - b^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

which gives the equation of a pair of chords of intersection of the circle and conic. If the conic is an ellipse, these chords never meet the curve in real points.

79. To find the invariant relation connecting a conic and the polar circle of the triangle formed by two tangents to the curve and their chord of contact.

If we make the equation (1) in the preceding example identical with

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0, \qquad (1)$$

we get $$k^2 - b^2 = \frac{a^2 + b^2 + \frac{c^2 y'^2}{b^2}}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}}, \quad k^2 - a^2 = \frac{a^2 + b^2 - \frac{c^2 x'^2}{a^2}}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}};$$

therefore $$\frac{a^2 \alpha^2}{k^2 - b^2} = \frac{x'^2}{a^2} \frac{\left(a^2 + b^2 + \frac{c^2 y'^2}{b^2}\right)}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}},$$

$$\frac{b^2 \beta^2}{k^2 - a^2} = \frac{y'^2}{b^2} \frac{\left(a^2 + b^2 - \frac{c^2 x'^2}{a^2}\right)}{\frac{x'^2}{a^2} + \frac{y'^2}{b^2}};$$

hence $$\frac{a^2 \alpha^2}{k^2 - b^2} + \frac{b^2 \beta^2}{k^2 - a^2} = a^2 + b^2. \qquad (2)$$

We can find this relation also as follows :—If two conics can be reduced to the forms $ax^2 + by^2 + cz^2 = 0$, $z^2 - xy = 0$, they must be connected by the invariant relation $\Theta\Theta' - \Delta\Delta' = 0$, or two roots of the equation $\Delta k^3 + \Theta k^2 + \Theta' k + \Delta' = 0$ must be equal with opposite signs. Expressing, then, that two roots of the equation at Ex. 1 (10) are connected by the relation $h_1^2 + h_2^2 = 0$, we obtain the relation (2); hence if the circle (1) cut orthogonally a fixed circle J, the locus of its centre is in general a cubic. If J passes through a pair of vertices of the curve, the locus reduces to a conic; for putting

$$r^2 = x^2 + y^2 - 2\beta y - a^2, \quad \text{or} \quad k^2 = 2\beta y + a^2,$$

we get from (2), after dividing by y,

$$a^2 x^2 = \left(a^2 + b^2 - \frac{b^2 y}{2\beta}\right)(c^2 - 2\beta y).$$

If the circle J is concentric with the curve, the locus is the conic obtained by considering k as a constant in the equation (2). This conic, it may be observed, touches the tangents of the given curve which are perpendicular to its asymptotes, and if $k^2 = \dfrac{a^4 + b^4 + a^2 b^2}{a^2 + b^2}$, will coincide with the given curve.

80. Tangents from a point P to the conic

$$U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

form with their chord of contact a triangle whose intersection of perpendiculars lies on U; show that P lies on the inverse of U with regard to its director circle.

Show also, in the same case, that the polar circle has double contact with the bicircular quartic

$$\left(x^2 + y^2 + \frac{a^4 + b^4 + a^2 b^2}{a^2 + b^2}\right)^2 = 4\,(a^2 x^2 + b^2 y^2).$$

81. If the polar circle of the triangle formed by the tangents from x, y to the conic $U \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, and their chord of contact, cut the director circle at an angle θ, show that

$$\cos^2 \theta = \frac{x^2 + y^2 - a^2 - b^2}{4a^2 b^2 (a^2 + b^2)\left(\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{c^4 x^2 y^2}{a^6 b^6}\right)}.$$

82. A point P moves along a fixed parallel to an axis of the conic; to find the envelope of the polar circle of the triangle formed by the tangents to the curve from P and their chord of contact.

Putting $x' = \alpha$ a constant in the equation, Ex. 78 (1), we see that the circle cuts orthogonally the fixed circle

$$x^2 + y^2 - \frac{a^2}{\alpha} x - b^2 = 0, \qquad (1)$$

and, eliminating y' between the co-ordinates of the centre, we find that the centre lies on the fixed conic

$$a^2 x^2 + b^2 y^2 + a^4 - b^4 - \left\{c^2 \alpha + \frac{a^4 + a^2 b^2}{\alpha}\right\} x = 0\,; \qquad (2)$$

hence the envelope is a bicircular quartic.

If we put $\alpha = \frac{a^2}{c}$ for the directrix, the circle (1) and the conic (2) have double contact with each other at two points on the directrix, and the envelope then breaks up into two circles. These two circles are imaginary for the ellipse.

If $a^2 (a^2 + b^2) - c^2 \alpha^2 = 0$, the equation of the circle becomes

$$x^2 + y^2 \pm \frac{2c}{a} \sqrt{(a^2 + b^2)}\, x + a^2 = 0, \qquad (3)$$

which is altogether fixed; hence we infer that the tangents to the curve from any point of either of the lines

$$a\sqrt{(a^2+b^2)} \pm cx = 0$$

form with their chord of contact a triangle whose intersection of perpendiculars is the fixed point

$$x' = \pm\frac{c}{a}\sqrt{(a^2+b^2)},\ y' = 0.$$

The rectangle under the segments of the perpendiculars is also given in this case, being equal to $\frac{-b^4}{a^2}$.

The circle (3) has double contact with the curve, and is always imaginary.

The same property holds, of course, also for the lines

$$b^2(a^2+b^2) + c^2y^2 = 0;$$

and the equation of the corresponding circle is, then,

$$x^2 + y^2 \pm 2\sqrt{(b^4-a^4)}\,\frac{y}{b} + b^2 = 0. \qquad (4)$$

These lines and the corresponding circles are real for a hyperbola whose director circle is imaginary.

It may be observed that the curve is its own reciprocal with respect to one of the circles (3) and (4); for the polar of any point x', y' with regard to the circle (3) is

$$xx' + yy' \pm \frac{c}{a}\sqrt{(a^2+b^2)}(x+x') + a^2 = 0,$$

and this line, subject to the condition

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0,$$

is a tangent to the curve. This is, of course, also evident from the fact that one tangent, and the point of contact of the other, are polar and pole with regard to the circle.

83. Tangents are drawn from any point of $xy - \alpha y - \beta x = 0$ to the equilateral hyperbola $x^2 - y^2 - a^2 = 0$; show that they form with their chord of contact a triangle whose polar circle cuts a fixed circle orthogonally, and has its centre on the equilateral hyperbola

$$x^2 - y^2 - 2\alpha x + 2\beta y = 0.$$

84. Show that the equation of the nine-point circle of the triangle formed by the tangents from x', y' to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

and their chord of contact is

$$2\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right)(x^2 + y^2) - \frac{xx'}{a^2}\left\{x'^2 + (2a^2 - b^2)\frac{y'^2}{b^2} + 3a^2 + b^2\right\}$$
$$- \frac{yy'}{b^2}\left\{y'^2 + (2b^2 - a^2)\frac{x'^2}{a^2} + 3b^2 + a^2\right\} + x'^2 + y'^2 + a^2 + b^2 = 0.$$

85. To find the locus of the vertex of a triangle formed by two tangents to a conic and their chord of contact, if the centre of the inscribed circle lies on the curve.

If α and β are the tangents, and γ their chord of contact, the equation of the conic must be capable of being written in the form $\alpha\beta - k\gamma^2 = 0$. But $\alpha = \beta = \gamma$ for the centre of the inscribed circle; hence we get $k = 1$. Substituting, now, for α, β, γ, $x \cos \alpha + y \sin \alpha - p$, &c., we must have $\alpha\beta - \gamma^2$ identical with $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$; hence, equating the coefficient of xy to zero, we see that the base and one of the bisectors of the vertical angle must make equal angles with the axis.

But $\frac{xx'}{a'^2} + \frac{yy'}{a'^2 - c^2} - 1 = 0$, being the tangent to a confocal conic through x', y', is a bisector of the vertical angle, and this line and $\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0$ must be equally inclined to the axis; thus we have

$$a'^2 = \frac{a^2(a^2 - b^2)}{a^2 + b^2},$$

or the locus is the confocal conic

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2 - b^2}{a^2 + b^2}. \qquad (1)$$

Since the equation $\alpha\beta \quad \gamma^2 = 0$ is satisfied by $\alpha = \beta = -\gamma$, we see that the centre of the circle exscribed to the base is also, in this case, situated on the curve.

The equation $\frac{a^2 x'}{x} - \frac{b^2 y'}{y} - c^2 = 0$ represents the equilateral hyperbola which passes through the feet of the normals from x', y' to the curve, and if the polar $\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0$ of x', y' cut this hyperbola orthogonally at x, y, we must have

$$x'^2 y^2 - y'^2 x^2 = 0.$$

Taking the factor $x'y + y'x = 0$, we find that x', y' must satisfy the equation

$$\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = \frac{c^2}{a^2 + b^2},$$

which coincides with the conic (1); hence we see that, in the above case, the base is normal to the equilateral hyperbola which passes through the feet of the normals to the curve from the vertex of the triangle.

86. In the preceding example show that the product of the diameters of the curve parallel to the tangents is equal to the square of the diameter parallel to the chord of contact.

87. From a point x', y' tangents are drawn to a conic, to find the co-ordinates of the focus of the parabola having double contact with the curve at their points of contact.

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ be the equation of the conic; then

$$(x'\lambda + y'\mu + \nu)^2 + k(a^2\lambda^2 + b^2\mu^2 - \nu^2) = 0$$

is the tangential equation of a conic having double contact with the curve at the points of contact of the tangents from x', y'; but for a parabola the coefficient of ν^2 must vanish, whence $k = 1$, and

$$(x'^2 + a^2)\lambda^2 + (y'^2 + b^2)\mu^2 + 2x'y'\lambda\mu + 2x'\nu\lambda + 2y'\mu\nu = 0$$

represents the required parabola; hence, to determine the foci, we have (*Conics*, Art. 279, Ex.)

$$2y'y - 2x'x + x'^2 - y'^2 + c^2 = 0, \quad x'y + y'x - x'y' = 0;$$

therefore $\quad 2x = x' \frac{(x'^2 + y'^2 + c^2)}{x'^2 + y'^2}, \; 2y = y' \frac{(x'^2 + y'^2 - c^2)}{x'^2 + y'^2}.$

From these expressions it follows that if the vertex of the triangle be fixed, and the conics belong to a confocal system, the foci of the parabolæ will remain fixed.

If x', y' lies on a line through the centre of a concentric circle, x, y lies on a confocal conic.

Given x, y, we determine x', y' as follows:—Let a confocal hyperbola be described through x, y, then the tangent to the hyperbola at x, y intersects its asymptotes in the corresponding positions of x', y'.

88. From a point x', y' tangents are drawn to a conic; prove that the centre of the equilateral hyperbola, having double contact with the curve at their points of contact, is the inverse point of $x'\ y'$ with regard to the director circle.

V.—CIRCLES HAVING DOUBLE CONTACT WITH THE CURVE.

89. There are two systems of circles having double contact with a conic, the chords of contact of each system being parallel to one of the axes of the curve.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is the equation of the curve, and $x-x'=0$ that of a chord of contact, the equation of the corresponding circle is

$$x^2+y^2-2e^2x'x+e^2x'^2-b^2=0\,; \qquad (1)$$

and if $y-y'=0$ is the chord of contact, the corresponding circle is

$$x^2+y^2+2c^2\frac{y'y}{b^2}-a^2-c^2\frac{y'^2}{b^2}=0. \qquad (2)$$

If α is the abscissa of the centre of the circle (1), and r its radius, we find

$$\frac{\alpha^2}{c^2}+\frac{r^2}{b^2}=1\,; \qquad (3)$$

and if β is the ordinate of the centre of (2), and r' its radius,

$$\frac{r'^2}{a^2}-\frac{\beta^2}{c^2}=1. \qquad (4)$$

We may conveniently satisfy the equation (3) by assuming

$$\alpha=c\cos\theta, \quad r=b\sin\theta.$$

From the equation (4) we see that the radius of a circle of the system (2) is in a constant ratio to the distance of its centre from either of the foci.

90. To find the differential equation in elliptic co-ordinates of the system of circles having double contact with a conic.

If we write the equation of the circle (1) in the preceding example in the form

$$x^2 + y^2 - 2cx \cos\theta + c^2 \cos^2\theta - b^2 \sin^2\theta = 0,$$

and transform this equation to elliptic co-ordinates by assuming

$$x^2 + y^2 = \mu^2 + \nu^2 - c^2, \quad cx = \mu\nu,$$

we get
$$\mu^2 + \nu^2 - 2\mu\nu \cos\theta - a^2 \sin^2\theta = 0,$$

which is equivalent to the relation

$$\cos^{-1}\frac{\mu}{a} \pm \cos^{-1}\frac{\nu}{a} = \theta; \qquad (1)$$

hence, by differentiation, we obtain

$$\frac{d\mu}{\sqrt{(a^2 - \mu^2)}} \pm \frac{d\nu}{\sqrt{(a^2 - \nu^2)}} = 0. \qquad (2)$$

From this equation we see at once that the two circles of the same system which pass through a point are equally inclined to the confocals through the point.

In a similar manner we find that

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = 0 \qquad (3)$$

is the differential equation of the second system of circles.

91. To find the angle between two circles of the same system which pass through a point.

If r_1, r_2 are the radii of the circles, and d the distance between their centres, we have

$$\tan^2 \tfrac{1}{2}\phi = \frac{d^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - d^2},$$

where ϕ is the angle sought. But

$$r_1 = b \sin \theta_1,\ r_2 = b \sin \theta_2,\ d = c(\cos \theta_1 - \cos \theta_2);$$

and from (1) in the preceding example we have

$$\mu = a \cos \tfrac{1}{2}(\theta_1 - \theta_2),\ \nu = a \cos \tfrac{1}{2}(\theta_1 + \theta_2); \tag{1}$$

hence
$$\tan \tfrac{1}{2}\phi = \sqrt{\left\{\frac{(a^2 - \mu^2)(c^2 - \nu^2)}{(\mu^2 - c^2)(a^2 - \nu^2)}\right\}}. \tag{2}$$

We could find this expression at once from the differential equation; for when two curves are represented by the differential equations $Pd\mu \pm Qd\nu = 0$, the angle ϕ between them is given by

$$\tan \tfrac{1}{2}\phi = \frac{Q}{P}\sqrt{\left(\frac{c^2 - \nu^2}{\mu^2 - c^2}\right)},$$

whence from Ex. 90 (2) we have the result already obtained.

From (2) we get

$$\cos \phi = \frac{b^2 x^2 + (a^2 + c^2) y^2 - b^2 c^2}{b^2 \rho\rho'},$$

and
$$\sin \phi = 2a \frac{cy}{b} \frac{\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}}{\rho\rho'},$$

where ρ, ρ' are the distances of x, y from the foci; hence, being given the angle between two circles of the system, the locus of their intersection is a curve of the fourth order of which the foci are nodes. If the angle is right, the locus is a concentric conic passing through the foci.

In a similar manner for the other system of circles we find

$$\tan \tfrac{1}{2}\phi = \frac{\nu\sqrt{(\mu^2 - a^2)}}{\mu\sqrt{(a^2 - \nu^2)}}, \tag{3}$$

and
$$\sin\phi = 2b\,\frac{cx}{a}\,\frac{\sqrt{\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)}}{\rho\rho'}$$
$$= \frac{cx}{a^2}\sin\alpha,$$

where α is the angle between the tangents to the curve from x, y (*Conics*, Art. 169, Ex. 3, and Art. 226, Ex. 12).

From these values it readily follows that the circles described through any point P on a directrix, to have double contact with a conic, are at right angles to the tangents from P.

92. If ψ is the angle which an external common tangent of two circles of the system makes with the axis, we have

$$\sin\psi = \frac{r_1 - r_2}{d} = \frac{b}{c}\,\frac{(\sin\theta_1 - \sin\theta_2)}{\cos\theta_1 - \cos\theta_2} = -\frac{b}{c}\cot\tfrac{1}{2}(\theta_1+\theta_2)$$
$$= -\frac{b\nu}{c\sqrt{(a^2-\nu^2)}}$$
from Ex. 91 (1);

hence, supposing the given curve to be an ellipse, the external common tangents are parallel to fixed lines when the intersection of the circle lies on a confocal hyperbola.

93. If n circles having double contact with a conic form with a single point of intersection of each a polygon, $n-1$ of whose vertices move along confocal conics, the n^{th} vertex will also move along a confocal conic.

If $\theta_1, \theta_2 \ldots \theta_n$ are the parameters of the n circles, we see from Ex. 91 (1) that we must have

$$\theta_1 - \theta_2 = \text{a constant}, \quad \theta_2 - \theta_3 = \text{a constant, \&c.},$$

whence $\theta_n - \theta_1$ is given, which proves the theorem. For a triangle the semiaxes μ_1, μ_2, μ_3 of the three confocal conics are connected by the relation

$$a^3 - a(\mu_1^2 + \mu_2^2 + \mu_3^2) + 2\mu_1\mu_2\mu_3 = 0.$$

94. If d is the distance between the centres of two circles of the same system, we have

$$d = c(\cos\theta_1 - \cos\theta_2) = -2c\sin\tfrac{1}{2}(\theta_1 - \theta_2)\sin\tfrac{1}{2}(\theta_1 + \theta_2)$$

$$= \frac{2c}{a^2}\sqrt{\{(a^2 - \mu^2)(a^2 - \nu^2)\}}$$

$$= \frac{2bc}{a}\sqrt{\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}.$$

From this expression we see that the theorem in the last example is also true if we substitute for confocal conics "concentric, similar, and similarly situated conics."

95. If A, B are the base angles of the triangle formed by the centres of two circles of the system at Ex. 89 (1), and a point of intersection, we have,

$$\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B = \frac{r_1 + r_2 - d}{r_1 + r_2 + d} = \frac{b\mu + c\sqrt{(a^2 - \mu^2)}}{b\mu - c\sqrt{(a^2 - \mu^2)}}$$

from Ex. 91 (1),

and
$$\frac{\tan\tfrac{1}{2}A}{\tan\tfrac{1}{2}B} = \frac{b\nu + c\sqrt{(a^2 - \nu^2)}}{b\nu - c\sqrt{(a^2 - \nu^2)}};$$

hence we see that either the product or ratio of the tangents of half the angles A and B is given, when the intersection of the circles lies on a confocal conic.

This property is, of course, also true for the system of circles which touch the curve externally (Ex. 89 (2)); and since these circles become tangents, when the curve degenerates into a parabola, we see that, if tangents to a parabola intersect on a confocal parabola, the product or ratio of the tangents of the halves of the angles which they make with the axis of the curve is constant.

96. Given the difference of the angles A and B in the preceding example, show that the locus of the intersection of the circles is an equilateral hyperbola passing through the foci.

97. If x, x' are the abscissae of the centres of similitude of two circles of the system at Ex. 89 (2), we have

$$x = \frac{r_1 x_2 + r_2 x_1}{r_1 + r_2} = c\,\frac{\cos \frac{1}{2}(\theta_1 - \theta_2)}{\cos \frac{1}{2}(\theta_1 + \theta_2)},$$

and $$x' = c\,\frac{\cos \frac{1}{2}(\theta_1 + \theta_2)}{\cos \frac{1}{2}(\theta_1 - \theta_2)};$$

therefore $xx' = c^2$, or the centres of similitude are harmonically conjugate with the foci.

For the second system of circles we find, similarly, $yy' = -c^2$, from which we see that the centres of similitude of two circles of this system subtend right angles at the foci.

98. If ϕ, ϕ' are the angles which the tangents from the foci to a circle of the system at Ex. 89 (1) make with the axis, we have

$$\sin \phi = \frac{r}{c + a} = \frac{b \sin \theta}{c\,(1 + \cos \theta)} = \frac{b}{c} \tan \tfrac{1}{2}\theta,$$

and $$\sin \phi' = \frac{b}{c} \cot \tfrac{1}{2}\theta;$$

therefore $$\sin \phi \sin \phi' = \frac{b^2}{c^2};$$

hence, if $e^2 < \frac{1}{2}$, one or other of the tangents is always imaginary.

99. If we seek the intersection of the circle

$$x^2 + y^2 + 2c^2 \frac{y'y}{b^2} - \left(a^2 + \frac{c^2 y'^2}{b^2}\right) = 0$$

with an asymptote of the curve, by putting

$$x = \rho \frac{a}{c}, \quad y = \rho \frac{b\sqrt{-1}}{c},$$

we get $$\rho^2 + 2c \frac{y'\sqrt{-1}}{b} \rho - a^2 - \frac{c^2 y'^2}{b^2} = 0;$$

and if ρ_1, ρ_2 are the roots of this equation, we have $\rho_1 - \rho_2 = 2a$; hence a circle having external double contact with a hyperbola cuts off from the asymptotes constant intercepts equal to the major axis of the curve.

100. The polar of x_1, y_1 with regard to the circle

$$x^2 + y^2 + 2 \frac{c^2}{b^2} y' y - a^2 - \frac{c^2}{b^2} y'^2 = 0,$$

being $$xx_1 + yy_1 + \frac{c^2}{b^2} y' (y + y_1) - a^2 - \frac{c^2}{b^2} y'^2 = 0,$$

its envelope, when the circle varies, is the parabola

$$(y - y_1)^2 + 4 \frac{a^2 b^2}{c^2} \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right) = 0. \quad (1)$$

Putting $y_1 = 0$, $x_1 = c$ for a focus, (1) becomes

$$y^2 = \frac{4b^2}{c}\left(\frac{a^2}{c} - x\right),$$

and the parabola then has its focus and directrix in common with the curve; hence we see that the feet of the perpendiculars from a focus on its polars, with regard to circles having external double contact with the curve, lie on the corresponding directrix.

101. Taking two circles of the system at Ex. 89 (1), namely,

$$x^2 + y^2 - 2e^2 x_1 x + e^2 x_1^2 - b^2 = 0, \quad x^2 + y^2 - 2e^2 x_2 x + e^2 x_2^2 - b^2 = 0,$$

we have for their intersection

$$x^2 + y^2 = b^2 + e^2 x_1 x_2, \quad 2x = x_1 + x_2;$$

therefore $$y^2 + (x \pm c)^2 = (a \pm ex_1)(a \pm ex_2);$$

hence we see that the square of the distance from a focus of a point of intersection of two circles having internal double contact with the curve is equal to the product of the distances from the same focus of their points of contact.

102. If two circles of the same system have double contact with a conic, show that the product of their radii is in a constant ratio to the product of the distances of a point of their intersection from the foci.

103. If two circles of the system at Ex. 89 (1) intersect in the points $x', \pm y'$, the equation of the circle passing through their points of contact is evidently

$$S \equiv x^2 + y^2 - 2e^2 x' x + x'^2 + y'^2 - 2b^2 = 0.$$

If S is fixed, and the conics form a confocal system, the locus of $x', \pm y'$ is a circle cutting S orthogonally.

If the points $x', \pm y'$ are fixed, and the conics form a confocal system, S has double contact with the Cartesian oval

$$(x^2 + y^2 + x'^2 + y'^2 + 2c^2)^2 - 16\, c^2 x' x = 0.$$

If the radius is given equal to r, the locus of $x', \pm y'$ is the conic

$$a^4 y^2 + b^2 (2a^2 - b^2)\, x^2 = a^4 (2b^2 - r^2).$$

If the tangents to S from x', $\pm y'$ contain a given angle, the polars of x', $\pm y'$, with regard to the curve, are touched by a confocal conic.

104. Tangents parallel to the line $y - mx = 0$ are drawn to the system of circles at Ex. 89 (1); show that their points of contact lie on

$$b^2x^2 + (c^2 + m^2a^2)\,y^2 + 2mb^2xy - b^2c^2 = 0.$$

Show that this conic passes through the foci and has double contact with the curve.

105. If $S \equiv x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$ represents a fixed circle, the radical axis of S and a circle of the system at Ex. 89 (1) is

$$2\alpha x + 2\beta y - k^2 - 2e^2x'x + e^2x'^2 - b^2 = 0\,;$$

and this line is a tangent to

$$e^2x^2 - 2\alpha x - 2\beta y + k^2 + b^2 = 0, \qquad (1)$$

which represents one of the parabolas passing through the points of intersection of S and the curve. For the second system of circles the radical axes touch the other parabola, which may be described through the same points.

When the circle S is touched by a circle of the system, the radical axis is the tangent at the point of contact; hence to construct the points where S is touched by four circles of the system, we draw the common tangents of S and the parabola (1), and then the points of contact of these tangents are the points required.

106. A circle S meets a conic in the points A, B, C, D. Four circles having internal double contact with the curve are described to touch S, and also four circles having external double contact; show that the chords of contact of the first

system intersect those of the second in the centres of the sixteen circles which may be described to touch the sides of the triangles *ABC*, *BCD*, &c. (see Ex. 11).

107. To find the locus of the centres of similitude of a fixed circle and circles having double contact with a conic.

The p and ω equation of a circle, whose centre is x', 0 and radius r, is evidently

$$p = r + x' \cos \omega. \qquad (1)$$

But for the system of circles at Ex. 89 (3) we may take

$$x' = c \cos \theta, \quad r = b \sin \theta;$$

also, if α, β are the perpendiculars from the foci on the line p, ω, we have

$$p = \tfrac{1}{2}(\alpha + \beta), \quad c \cos \omega = \tfrac{1}{2}(\alpha - \beta);$$

hence (1) becomes

$$\alpha \tan \tfrac{1}{2}\theta + \beta \cot \tfrac{1}{2}\theta - 2b = 0, \qquad (2)$$

the envelope of which is $\alpha\beta - b^2 = 0$, as it ought to be.

If we now write the tangential equation of the fixed circle in the form $\Sigma' \equiv \gamma^2 - k^2 = 0$, the equation

$$\alpha \tan \tfrac{1}{2}\theta + \beta \cot \tfrac{1}{2}\theta \pm \frac{2b}{k}\gamma = 0 \qquad (3)$$

will represent one of the centres of similitude of Σ' and the circle (2). The envelope of (3) with regard to θ will then give the locus required, viz.,

$$\alpha\beta - \frac{b^2}{k^2}\gamma^2 = 0, \quad \text{or} \quad \alpha\beta - b^2 + \frac{b^2}{k^2}(\gamma^2 - k^2) = 0, \quad (4)$$

which represents a conic passing through the foci and touching the common tangents of Σ' and the given curve.

If the equation of Σ in Cartesian co-ordinates is

$$(x - x')^2 + (y - y')^2 - k^2 = 0,$$

the locus is found to be

$$\frac{k^2}{b^2} y^2 + \frac{1}{c^2}(x'y - y'x)^2 - (y - y')^2 = 0.$$

We see then that the locus will become a circle if

$$x' = 0, \quad \frac{k^2}{b^2} - \frac{y'^2}{c^2} = 1,$$

in which case Σ has double contact with the confocal conic

$$\frac{x^2}{b^2} + \frac{y^2}{2b^2 - a^2} = 1.$$

The locus (4) evidently passes through the four points on Σ, where it is touched by circles of the system (2).

108. If we take two circles of the system (2) in the preceding example, namely,

$$\alpha \tan \tfrac{1}{2}\theta_1 + \beta \cot \tfrac{1}{2}\theta_2 - 2b = 0, \quad \alpha \tan \tfrac{1}{2}\theta_2 + \beta \cot \tfrac{1}{2}\theta_2 - 2b = 0,$$

we find that their common tangents satisfy the equations

$$\tan \tfrac{1}{2}(\theta_1 + \theta_2) = \frac{2b}{\alpha - \beta}, \quad \tan \tfrac{1}{2}(\theta_1 - \theta_2) = \frac{2\sqrt{(b^2 - \alpha\beta)}}{\alpha + \beta};$$

hence, from Ex. 91 (1), we see that, if the common tangents are parallel to a given line, or touch a concentric, similar and similarly situated conic, the intersection of the circles will lie on a fixed confocal conic.

109. Through a point P, external to a conic, two real circles are described to have double contact with the curve; if a' is half the axis major of the confocal conic touching the common tangents of the circles, and ϕ is the angle between

the tangents from P to the curve, show that $\sin \frac{1}{2}\phi = \frac{a}{a'}$, where a is half the axis major of the curve.

110. A right line touches two circles having double contact with a conic; to show that its points of contact with them lie on the same concentric, similar and similarly situated conic.

Let $x\cos\omega + y\sin\omega - p = 0$, $(x - \cos\theta)^2 + y^2 - b^2\sin^2\theta = 0$, be the equations of the line and one of the circles, respectively; and let x, y be the co-ordinates of their point of contact; then

$$x = c\cos\theta + b\sin\theta\cos\omega, \quad y = b\sin\theta\sin\omega, \tag{1}$$

$$p = b\sin\theta + c\cos\theta\cos\omega; \tag{2}$$

hence

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2}\{c^2\cos^2\theta + a^2\sin^2\theta + 2bc\sin\theta\cos\theta\cos\omega - c^2\sin^2\theta\cos^2\omega\}$$

equal to, from (2), $\dfrac{p^2 + c^2\sin^2\omega}{a^2} = \dfrac{a'^2}{a^2}$,

where a' is half the axis major of the confocal conic touching the line. The conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a'^2}{a^2}, \tag{3}$$

therefore, passes through the points where the line is touched by both the circles.

We can arrive at this result by means of the differential equation of the system of circles. From Ex. 90 (2) we have for the circles

$$\frac{d\mu}{\sqrt{(a^2 - \mu^2)}} \pm \frac{d\nu}{\sqrt{(a^2 - \nu^2)}} = 0. \tag{4}$$

Now, it is not difficult to see that

$$\frac{d\mu}{\sqrt{\{(\mu^2 - a'^2)(\mu^2 - c^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a'^2 - \nu^2)(c^2 - \nu^2)\}}} = 0 \quad (5)$$

is the differential equation of all the lines touching the confocal conic $\frac{x^2}{a'^2} + \frac{y^2}{a'^2 - c^2} - 1 = 0$. But when one of the lines touches one of the circles, $\frac{d\mu}{d\nu}$ must have the same value at the point of contact; hence, from (4) and (5) we have

$$\frac{(a^2 - \mu^2)}{(\mu^2 - a'^2)(\mu^2 - c^2)} - \frac{(a^2 - \nu^2)}{(a'^2 - \nu^2)(c^2 - \nu^2)} = 0;$$

or, dividing by $\mu^2 - \nu^2$,

$$(a^2 - \mu^2)(a^2 - \nu^2) - b^2(a^2 - a'^2) = 0,$$

which is equivalent to the relation (3).

Putting $a' = c$ in (3), we see that the points of contact of the tangents from the foci to the system of circles lie on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = e^2.$$

111. To find the distance δ between the centres of two circles of the system which touch a given chord of the curve.

Let the equation of the chord be

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta) = 0,$$

and that of the circle

$$(x - c\cos\theta)^2 + y^2 - b^2\sin^2\theta = 0;$$

then, expressing that the chord touches the circle, we get

$$\{\cos\tfrac{1}{2}(\alpha - \beta) - e\cos\theta\cos\tfrac{1}{2}(\alpha+\beta)\}^2 - \sin^2\theta\{1 - e^2\cos^2\tfrac{1}{2}(\alpha+\beta)\} = 0,$$

or
$$\cos^2\theta - 2e\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)\cos\theta$$
$$+ \cos^2\tfrac{1}{2}(\alpha-\beta) - \{1 - e^2\cos^2\tfrac{1}{2}(\alpha+\beta)\} = 0;$$

whence we have
$$\cos\theta_1 - \cos\theta_2 = 2\sin\tfrac{1}{2}(\alpha-\beta)\sqrt{\{1 - e^2\cos^2\tfrac{1}{2}(\alpha+\beta)\}}$$
$$= \frac{2}{a}b'\sin\tfrac{1}{2}(\alpha-\beta);$$

but if d is the length of the chord,
$$d = 2b'\sin\tfrac{1}{2}(\alpha-\beta);$$

therefore
$$\delta = c(\cos\theta_1 - \cos\theta_2) = e d.$$

If the curve is a hyperbola, the two circles having external contact with the curve will also be real, and for them we shall have
$$\delta' = \sqrt{\left(\frac{b^2 - a^2}{b^2}\right)}\, d.$$

112. Tangents are drawn from the foci to circles having external double contact with a hyperbola; show that their points of contact lie on the asymptotes.

113. Through the centre of a circle having double contact with a conic tangents are drawn to a confocal conic; to show that they meet the circle on two chords of intersection of the conics.

Expressing that the curves represented by the differential equations (4) and (5) in Ex. 110 cut at right angles, we obtain
$$\frac{(a^2-\mu^2)(\mu^2-a'^2)}{\mu^2-c^2} - \frac{(a^2-\nu^2)(a'^2-\nu^2)}{c^2-\nu^2} = 0;$$

or, dividing by $\mu^2 - \nu^2$,
$$(\mu^2 - c^2)(c^2 - \nu^2) + b^2 b'^2 = 0,$$

which, transformed to Cartesian co-ordinates, becomes

$$c^2 y^2 + b^2 b'^2 = 0;$$

but this represents the two chords of the conics which are parallel to the transverse axis.

114. Through the centre of a circle having external double contact with a conic lines are drawn to the foci; show that they meet the circle on the tangents to the curve at the extremities of the transverse axis.

115. Tangents are drawn from a focus to a circle having external double contact with a hyperbola; show that they contain an angle equal to that between the asymptotes of the curve.

116. To find the condition that four circles of the system should be all touched by the same circle.

If we express that the circle

$$(x - c\cos\theta)^2 + y^2 - b^2 \sin^2\theta = 0$$

touches the circle

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0,$$

we obtain $$(\alpha - c\cos\theta)^2 + \beta^2 - (r \pm b\sin\theta)^2 = 0. \qquad (1)$$

Now if we put

$$c\cos\theta = x, \quad b\sqrt{-1}\sin\theta = y,$$

this relation may be replaced by the equations

$$\frac{x^2}{c^2} - \frac{y^2}{b^2} = 1, \qquad (2)$$

$$(x - \alpha)^2 + (y - r\sqrt{-1})^2 + \beta^2 = 0; \qquad (3)$$

and thus we see that we have to find the condition that four points on the conic (2) should lie on the circle (3); but this is known to be

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0, \quad \text{or} \quad 2m\pi. \qquad (4)$$

When this condition is satisfied, a confocal conic passing through the intersection of one pair of the circles will also pass through the intersection of the remaining pair (see Ex. 91 (1)).

If four circles having double contact with a parabola are all touched by the same circle, we find that the algebraic sum of their radii is equal to zero.

117. Let us suppose three of the circles in the preceding example to coincide, then we see from (4) that, if the osculating circles at the points $\pm\,\theta$ on the curve are touched by a circle of the system whose parameter is θ', we must have

$$3\theta + \theta' = 0, \quad \text{or} \quad 2\pi\,;$$

hence three pairs of osculating circles can be described to touch a given circle of the system; and if circles of the system be described at the points of contact of these osculating circles, their intersection will lie on the confocal conic

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 4c^2} = \tfrac{1}{4} \quad \text{(see Ex. 93).}$$

We can also show that—(1) the confocal conic passing through the points of contact of one of the three latter circles will also pass through the intersection of the remaining pair; (2) the algebraic sum of the radii of these circles vanishes; (3) the centroid of their centres coincides with the centre of the curve.

118. To find the equations of the circles which touch three given circles of the system.

From Ex. 116 (2) and (3), we see that the problem is the same as that of finding the equation of the circle passing through three points on the conic $\dfrac{x^2}{c^2} - \dfrac{y^2}{b^2} = 1$; hence from

Ex. 1 (7), (8), and (9), if the equation of the circle sought is

$$x^2 + y^2 - 2Ax - 2By + C = 0,$$

we obtain

$$A = \frac{a^2}{c} \cos \tfrac{1}{2}(\theta_1 + \theta_2) \cos \tfrac{1}{2}(\theta_2 + \theta_3) \cos \tfrac{1}{2}(\theta_3 + \theta_1),$$

$$B = \pm \frac{1}{bc} \sqrt{\{c^2 - a^2 \cos^2 \tfrac{1}{2}(\theta_1 + \theta_2)\} \{c^2 - a^2 \cos^2 \tfrac{1}{2}(\theta_2 + \theta_3)\}}$$
$$\times \{c^2 - a^2 \cos^2 \tfrac{1}{2}(\theta_1 + \theta_2)\},$$

$$C = \tfrac{1}{2} a^2 \{\cos(\theta_1 + \theta_2) + \cos(\theta_2 + \theta_3) + \cos(\theta_3 + \theta_1)\} - \tfrac{1}{2}(a^2 - 2b^2);$$

and, if R is the radius of this circle,

$$R = \frac{a^2}{b} \sin \tfrac{1}{2}(\theta_1 + \theta_2) \sin \tfrac{1}{2}(\theta_2 + \theta_3) \sin \tfrac{1}{2}(\theta_3 + \theta_1).$$

The equations of the three other pairs of circles will evidently be obtained by changing the sign of one of the angles θ_1, θ_2, θ_3.

119. To find the angle between a tangent to a conic and a circle having external double contact with the curve.

Let the equation of the tangent be

$$x \cos \omega + y \sin \omega = p = \sqrt{(a^2 \cos^2 \omega + b^2 \sin^2 \omega)},$$

and that of the circle

$$x_2 + (y - \beta)^2 = r^2 = a^2 \left(1 + \frac{\beta^2}{c^2}\right) \quad \text{(see Ex. 89 (4))}.$$

Then, expressing that the perpendicular from the centre of the circle on the tangent is equal to $r \cos \theta$, we get

$$r \cos \theta = \beta \sin \omega - p, \qquad (1)$$

where θ is the angle sought. But, putting

$$p = a \sin \alpha, \quad \beta = c \cot \phi,$$

we find $$c \sin \omega = a \cos \alpha, \quad r = \frac{a}{\sin \phi};$$

hence (1) gives at once

$$\theta = \alpha + \phi. \qquad (2)$$

Taking two circles and the same tangent we have

$$\theta_1 - \theta_2 = \phi_1 - \phi_2\,;$$

hence we see that the tangents to a conic meet two fixed circles, having external double contact with the curve at angles whose sum or difference is constant and equal to the angle subtended at one of the foci by the centres of the circles.

Taking two tangents and the same circle, we have

$$\theta_1 - \theta_2 = \alpha_1 - \alpha_2\,;$$

hence we see that a variable circle having double contact with a conic meets two fixed tangents to the curve at angles whose sum or difference is constant. This constant, it is easy to see, is equal to half the difference of the angles subtended by the foci at the points of contact of the tangents.

120. To express the angle θ in the preceding example in terms of the co-ordinates x, y of the intersection of the tangent and circle.

This may be most readily effected by means of the differential equations of the circle and tangent in elliptic co-ordinates. These equations are

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = 0, \quad (1)$$

$$\frac{d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = 0, \quad (2)$$

respectively (see Ex. 90 (3)).

But when two curves are represented by the differential equations in elliptic co-ordinates,

$$Pd\mu + Qd\nu = 0, \quad P'd\mu + Q'd\nu = 0,$$

the angle θ between them is given by

$$\tan\theta = \frac{(PQ' - P'Q)\sqrt{\{(\mu^2 - c^2)(c^2 - \nu^2)\}}}{PP'(\mu^2 - c^2) + QQ'(c^2 - \nu^2)};$$

hence, from (1) and (2) we have

$$\tan\theta = \frac{\sqrt{\{(\mu^2 - a^2)(a^2 - \nu^2)\}}}{a^2 \pm \mu\nu},$$

and, therefore,
$$\cos\theta = \frac{a^2 \pm \mu\nu}{a(\mu \pm \nu)}.$$

Transforming this expression to Cartesian co-ordinates, we get

$$\cos\theta = \frac{a + ex}{\rho}, \text{ or } \frac{a - ex}{\rho'},$$

where ρ, ρ' are the distances of x, y from the foci; θ, therefore, is equal to half the angle which the points of contact of the tangents from x, y subtend at one of the foci (Salmon's *Conics*, Art. 121).

121. Two circles are described through a point x, y to have external double contact with a conic; to find the angle subtended by their centres at a focus.

A circle having double contact with the curve being written in the form

$$\beta^2 + \frac{2b^2}{c^2}\beta y + \frac{b^2}{c^2}(x^2 + y^2 - a^2) = 0,$$

we have, for the two circles of the system which pass through x, y,

$$\beta_1\beta_2 = \frac{c^2}{b^2}(a^2 - x^2 - y^2),\ \beta_1 - \beta_2 = \frac{2ac}{b}\sqrt{\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)} = \frac{2ac}{b}\sqrt{S};$$

but, if ϕ is the angle sought, we have

$$\tan\phi = \frac{c(\beta_1 - \beta_2)}{c^2 + \beta_1\beta_2} = \frac{2ab\sqrt{S}}{a^2 + b^2 - x^2 - y^2},$$

from which we see that ϕ is equal to the angle between the tangents drawn to the curve from x, y (Salmon's *Conics*, Art. 169, Ex. 3).

This result might also be readily deduced from the relation (2) Ex. 119.

122. Two circles having external double contact with a conic are described to touch a tangent to a confocal conic; to show that the angle subtended by their centres at a focus is constant.

Expressing that the circle

$$x^2 + (y - \beta)^2 - r^2 = 0$$

has double contact with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and touches the line

$$x\cos\omega + y\sin\omega - p = 0,$$

we obtain $$\frac{r^2}{a^2} - \frac{\beta^2}{c^2} = 1,\quad \beta\sin\omega - p = r.$$

Eliminating r between these equations, we have

$$\beta^2(a^2 - c^2\sin^2\omega) + 2c^2p\sin\omega\beta + c^2(a^2 - p^2) = 0;$$

hence, if β_1, β_2 are the roots of this equation, we find

$$\beta_1 - \beta_2 = \frac{2ac\sqrt{(p^2 + c^2\sin^2\omega - a^2)}}{a^2 - c^2\sin^2\omega}, \qquad (1)$$

$$\beta_1\beta_2 = \frac{c^2(a^2 - p^2)}{a^2 - c^2\sin^2\omega}. \qquad (2)$$

But if ϕ is the angle sought,

$$\tan\phi = \frac{c(\beta_1 - \beta_2)}{c^2 + \beta_1\beta_2} = \frac{2a\sqrt{(p^2 + c^2\sin^2\omega - a^2)}}{2a^2 - p^2 - c^2\sin^2\omega}$$

from (1) and (2),

$$= \frac{2a\sqrt{(a'^2 - a^2)}}{2a^2 - a'^2}, \qquad (3)$$

where a' is half the axis major of the confocal conic touching the line. The relation (3) may be written in the simpler form

$$\cos\tfrac{1}{2}\phi = \frac{a}{a'}.$$

As a particular case we have:—The circles drawn through the foci to touch a variable tangent to a conic cut each other under a constant angle.

123. A circle having internal double contact with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

cuts orthogonally a circle having internal double contact with the confocal conic

$$\frac{x^2}{a'^2} + \frac{y^2}{a'^2 - c^2} - 1 = 0;$$

show that the locus of their intersection is

$$x^2 + \left(1 + \frac{c^2}{b^2} + \frac{c^2}{b'^2}\right)y^2 - c^2 = 0.$$

If both the circles have external double contact, show that the locus is

$$x^2\left(\frac{c^2}{a^2}+\frac{c^2}{a'^2}-1\right)-y^2-c^2=0.$$

124. To find the envelope of the tangents to the circles having double contact with a conic at the points where these circles are intersected by a fixed tangent to the curve.

Let the equation of one of the circles be

$$S \equiv x^2+y^2-2\beta y-a^2-\frac{b^2}{c^2}\beta^2=0, \qquad (1)$$

and that of the fixed tangent

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1=0, \qquad (2)$$

then we may write $S'=0$ in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1-\frac{c^2}{a^2b^2}\left(y+\frac{b^2}{c^2}\beta\right)^2=0\,;$$

but from (2) we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\left(\frac{x}{a}\sin\phi-\frac{y}{b}\cos\phi\right)^2;$$

therefore $$\frac{x}{a}\sin\phi-\frac{y}{b}\cos\phi\pm\frac{c}{ab}\left(y+\frac{b^2}{c^2}\beta\right)=0\,; \qquad (3)$$

hence, if x', y' is one of the points of intersection of (1) and (2), we find from (2) and (3)

$$\frac{x'}{a}=\frac{\cos\phi\pm\left(e+\dfrac{b\beta}{ac}\sin\phi\right)}{1\pm e\cos\phi}, \quad \frac{y'}{b}=\frac{\sin\phi\mp\dfrac{b\beta}{ac}\cos\phi}{1\pm e\cos\phi}.$$

The tangent to S at x', y' is, then,

$$ax\left\{\cos\phi \pm \left(e + \frac{b\beta}{ac}\sin\phi\right)\right\} + b\,(y-\beta)\left\{\sin\phi \mp \frac{b\beta}{ac}\cos\phi\right)$$
$$-\left(\frac{b^2}{c^2}\beta^2 + \beta y + a^2\right)(1 \pm e\cos\phi) = 0,$$

which, when β varies, touches one or other of two parabolae.

125. Let

$$S \equiv (x-\alpha)^2 + y^2 - r^2 = 0$$

be a circle having double contact with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and
$$S' \equiv x^2 + (y-\beta)^2 - r'^2 = 0$$

a circle having double contact with the confocal conic

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0;$$

then from (3) and (4), Ex. 89, we have

$$\frac{r^2}{b^2} + \frac{\alpha^2}{c^2} = 1, \ (1) \qquad \frac{r'^2}{a'^2} - \frac{\beta^2}{c^2} = 1; \ (2)$$

therefore
$$\frac{r'^2}{a'^2} - \frac{r^2}{b^2} = \frac{\alpha^2 + \beta^2}{c^2} = \frac{r^2 + r'^2 - 2rr'\cos\phi}{c^2},$$

where ϕ is the angle at which S and S' intersect.

Hence, when ϕ is given, the ratio of r to r' has one or other of two constant values.

Putting $r = nr'$ we get from (1) and (2)

$$b^2\alpha^2 + n^2a'^2\beta^2 = c^2(b^2 - n^2a'^2). \qquad (3)$$

Now if the line

$$\frac{x}{\alpha} + \frac{y}{\beta} - 1 = 0$$

is a normal to the conic

$$\frac{x^2}{l^2} + \frac{y^2}{m^2} = 1,$$

we have $$l^2 \alpha^2 + m^2 \beta^2 = (l^2 - m^2)^2.$$

We see thus that, when S and S' cut each other under a constant angle, the line joining their centres is normal to

$$\frac{x^2}{b^2} + \frac{y^2}{n^2 a'^2} - \frac{c^2}{b^2 - n^2 a'^2} = 0,$$

which represents one of two conics confocal with the given ones.

If x, y are the co-ordinates of a centre of similitude of S and S', we find, in the same case,

$$b^2 x^2 + a'^2 y^2 = c^2 \frac{(b^2 - n^2 a'^2)}{(n \pm 1)^2},$$

which represents one of four concentric conics.

126. If the circles S and S' in the preceding example touch one another, show that their point of contact lies on the conic

$$\frac{x^2}{a'^2} + \frac{y^2}{b^2} - 1 = 0.$$

If S and S' cut orthogonally, show that they intersect on

$$\frac{x^2}{a^2 a'^2} + \frac{y^2}{b^2 b'^2} = 0.$$

127. To find the orthogonal trajectory of the system of circles having double contact with a conic.

We have seen in Ex. 90 that the differential equation of the system having external contact with the curve is

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - c^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2 - \nu^2)(c^2 - \nu^2)\}}} = 0.$$

The differential equation of the trajectory is, therefore,

$$\frac{d\mu}{\mu}\sqrt{\left(\frac{\mu^2 - a^2}{\mu^2 - c^2}\right)} \mp \frac{d\nu}{\nu}\sqrt{\left(\frac{a^2 - \nu^2}{c^2 - \nu^2}\right)} = 0.$$

Taking the lower sign and integrating, we obtain

$$\log\{a\sqrt{(\mu^2 - c^2)} + c\sqrt{(\mu^2 - a^2)}\}\{a\sqrt{(c^2 - \nu^2)} + c\sqrt{(a^2 - \nu^2)}\} - \log\mu\nu$$
$$- e\log\{\sqrt{(\mu^2 - a^2)} + \sqrt{(\mu^2 - c^2)}\}\{\sqrt{(a^2 - \nu^2)}$$
$$+ \sqrt{(c^2 - \nu^2)}\} = \text{a constant}.$$

If, then, the eccentricity of the curve is equal to the ratio of two integers, the trajectory will be algebraic. For the system of circles which have internal contact with the curve the trajectory is transcendental when the curve is an ellipse, and algebraic when the curve is a hyperbola whose eccentricity is equal to $\frac{m}{\sqrt{(m^2 - n^2)}}$, where m and n are integers.

128. A variable circle has double contact with a conic; show that the tangents drawn through one of the points of contact to a fixed confocal conic intercept on the circle segments of given length.

129. Two circles of different systems have double contact with a conic; to show that the intersection of their chords of contact is a limiting point of the circles.

Taking the equations (1) and (2) in Ex. 89, we may write

$$S \equiv x^2 + y^2 - 2e^2x'x + e^2x'^2 - b^2,$$

$$S \equiv x^2 + y^2 + 2c^2\frac{y'y}{b^2} - a^2 - c^2\frac{y'^2}{b^2};$$

hence $$a^2S - b^2S' \equiv (a^2 - b^2)\{(x - x')^2 + (y - y')^2\},$$

which proves the result stated.

130. Four circles having double contact with a conic are described to cut a circle J orthogonally; show that their chords of contact form a rectangle inscribed in J so that each pair of opposite vertices are conjugate with respect to the curve.

131. Show that the second limiting point of the circles S and S' in Ex. 129 is the foot of the perpendicular from the point x', y' on its polar with regard to the curve.

132. To find the length of a common tangent of two circles having double contact with a hyperbola, the circles belonging to different systems.

If the circles

$$(x-\alpha)^2 + y^2 - r^2 = 0, \quad x^2 + (y-\beta)^2 - r'^2 = 0,$$

have double contact with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have from (3) and (4), Ex. 89,

$$\frac{\alpha^2}{c^2} + \frac{r^2}{b^2} = 1, \qquad \frac{r'^2}{a^2} - \frac{\beta^2}{c^2} = 1;$$

therefore $$\frac{r'^2}{a^2} - \frac{r^2}{b^2} = \frac{\alpha^2 + \beta^2}{c^2} = \frac{d^2}{c^2},$$

where d is the distance between the centres of the circles;

hence $$d^2 = e^2 r'^2 + \frac{e^2}{e^2-1} r^2. \qquad (1)$$

But, if t is the length of the common tangent,

$$t^2 = d^2 - (r \pm r')^2$$
$$= (e^2-1) r'^2 + \frac{r^2}{e^2-1} \mp 2rr' \text{ from (1)};$$

therefore $$t = \sqrt{(e^2-1)}\, r' \mp \frac{r}{\sqrt{(e^2-1)}}, \qquad (2)$$

the two signs corresponding to the internal and external common tangents; hence, if one circle remains fixed, while the other varies, the sum or difference of their external and internal common tangents will remain constant.

133. Let S be a variable circle having double contact with a hyperbola, and S_1, S_2 two fixed circles of a different system having double contact with the curve; then, if t_1 is a common tangent of S and S_1, and t_2 of S and S_2, show that $t_1 - t_2 =$ a constant.

134. A circle S whose centre is P touches the sides of a triangle inscribed in a hyperbola. If circles S_1, S_2 be described to have double contact with the curve, so that their chords of contact intersect in P, show that the radius of S is in a constant ratio to a common tangent of S_1 and S_2 (see Ex. 10, (3)).

135. A conic has double contact with two fixed circles; to find the locus of the foci.

If the circles belong to different systems, we see from Ex. 132, (1), that the eccentricity of the curve has one or other of two constant values. Hence, it readily follows, that the foci lie on one or other of two circles concentric with the circle which has external contact with the curve.

If both the circles have external contact with the curve, we see from Ex. 89, (4), that the distances of a focus from the centres of the circles are to one another as the radii of the circles. The foci, therefore, in this case, lie on a circle passing through the intersection of the given circles.

The complete locus, then, consists of five circles, besides the line joining the centres of the given circles.

136. If two circles are connected by the relation

$$d^2 = 2(r^2 + r'^2),$$

show that an infinite number of equilateral hyperbolae can be described to have double contact with them.

137. A hyperbola has double contact with two fixed circles; to find the envelope of the asymptotes. Suppose the circles to belong to different systems, then if A, B are their centres, and C is the centre of the curve, ACB is a right angle, and because the eccentricity is given (see Ex. 132, (1)) the asymptotes are inclined to CA and CB at constant angles. The asymptotes, therefore, pass through fixed points on the circle described on AB as diameter.

If the circles belong to the same system, an asymptote cuts off equal intercepts on the circles (see Ex. 99). The envelope then is easily proved to be a parabola of which the middle point of the centres of the circles is the focus.

138. A conic has double contact with two fixed circles, the circles belonging to different systems; to find the envelope of the directrices.

The equation

$$x^2 + y^2 - 2ax + k^2 - e^2 (x \cos\theta + y \sin\theta)^2 = 0, \qquad (1)$$

where θ is variable, evidently represents a system of conics having double contact with the fixed circles

$$x^2 + y^2 - 2ax + k^2 = 0, \quad (1 - e^2)(x^2 + y^2) - 2ax + k^2 = 0\,;$$

and since the chords of contact, namely,

$$x \cos\theta + y \sin\theta = 0, \quad x \sin\theta - y \cos\theta = 0,$$

are at right angles to each other, the circles are of different systems.

Writing the equation (1) in the form

$$x^2 + y^2 - 2ax + k^2 + e^2p^2 - 2e^2p\,(x \cos\theta + y \sin\theta)$$
$$= e^2 (x \cos\theta + y \sin\theta - p)^2,$$

we see that $$x \cos\theta + y \sin\theta - p = 0$$

will represent a directrix of the curve when the left-hand member of this equation represents the square of the distance from a focus. This condition gives

$$e^2 (1 - e^2) p^2 - 2e^2 ap \cos\theta + k^2 - a^2 = 0,$$

showing that the directrix touches a conic of which the origin is a focus.

139. To find the envelope of the director circles of the system of conics in the preceding example.

The conic being written in the form (1) in the preceding example, the equation of the director circle is

$$(1 - e^2)(x^2 + y^2) - 2ax (1 - e^2 \sin^2\theta) - 2e^2 ay \sin\theta \cos\theta$$
$$+ k^2 (2 - e^2) - a^2 = 0 \qquad (\textit{Conics}, \text{Art. } 294).$$

The envelope is, therefore,

$$\{(1 - e^2)(x^2 + y^2) - (2 - e^2) ax + k^2 (2 - e^2) - a^2\}^2 - e^4 a^2 (x^2 + y^2) = 0,$$

which represents a Cartesian oval of which the origin is a focus.

140. A conic has double contact with two circles, the circles belonging to the same system; to find the envelope of the director circle.

If α, β are the perpendiculars from the centres of the circles on a line, the equation

$$\frac{\alpha^2}{r^2} + \frac{\beta^2}{r'^2} + \frac{2\alpha\beta \cos\theta}{rr'} - \sin^2\theta = 0 \qquad (1)$$

evidently represents tangentially a conic having double contact with the circles

$$\alpha^2 - r^2 = 0, \quad \beta^2 - r'^2 = 0.$$

Now if the co-ordinates of the centres of the circles are $\pm c$, 0, we have

$$\alpha = \frac{\lambda c + \nu}{\sqrt{(\lambda^2 + \mu^2)}}, \qquad \beta = \frac{-\lambda c + \nu}{\sqrt{(\lambda^2 + \mu^2)}},$$

and (1) becomes

$$\frac{(\lambda c + \nu)^2}{r^2} + \frac{(\lambda c - \nu)^2}{r'^2} - \frac{2(\lambda^2 c^2 - \nu^2)\cos\theta}{rr'} - \sin^2\theta\,(\lambda^2 + \mu^2) = 0\,;$$

hence (*Conics*, Art. 294) the equation of the director circle is

$$(r^2 + r'^2 - 2rr'\cos\theta)(x^2 + y^2) - 2c\,(r'^2 - r^2)\,x$$
$$+ c^2(r^2 + r'^2 - 2rr'\cos\theta) - 2r^2r'^2\sin^2\theta = 0\,;$$

the envelope is, therefore,

$$(x^2 + y^2 - c^2)^2 - 2r^2\{y^2 + (x + c)^2\} - 2r'^2\{y^2 + (x - c)^2\} + 4r^2r'^2 = 0,$$

which represents a Cartesian oval, of which the origin is the triple focus, and, it can be shown, the centres of similitude of the circles are single foci.

141. A conic has double contact with two fixed circles; to find the locus of the points through which two curves of the system cut orthogonally.

If the circles belong to different systems, we write the equation of the conic in the form (1), Ex. 138. Expressing, then, that the two conics corresponding to the parameters θ_1, θ_2 cut orthogonally, we get

$$e^2 S\cos(\theta_1 - \theta_2) + (x - a)^2 + y^2 - e\sqrt{S}\,\{(x - a)(\cos\theta_1 + \cos\theta_2)$$
$$+ y\,(\sin\theta_1 + \sin\theta_2)\} = 0,$$

where $$S \equiv x^2 + y^2 - 2ax + k^2.$$

But $$\cos(\theta_1 - \theta_2) = \frac{2S}{x^2 + y^2} - e^2, \quad \cos\theta_1 + \cos\theta_2 = \frac{2x\sqrt{S}}{e\,(x^2 + y^2)}.$$

We find thus, finally,

$$(x^2+y^2-2ax+k^2)\{(1-e^2)(x^2+y^2)-2ax+k^2\}+(ax-k^2)^2+a^2y^2=0,$$

which represents a bicircular quartic, of which the centres of the fixed circles are double foci.

If the circles belong to different systems, the locus is easily proved to be the circle described on the line joining the centres of similitude of the circles as diameter.

142. Given three circles with their centres on a line, there is, in general, a single conic having double contact with them.

Since the tangent from any point on the conic to one of the circles is in a constant ratio to the perpendicular on the chord of contact, it can easily be seen that

$$l\sqrt{S_1}+m\sqrt{S_2}+n\sqrt{S_3}=0, \qquad (1)$$

where S_1, S_2, S_3 are the squares of the tangents to the circles, and l, m, n the distances between their centres, represents the conic. This equation (1) when cleared of radicals is of the second degree, the terms of higher orders vanishing identically.

In a similar manner, if Σ_1, Σ_2, Σ_3 are the squares of the intercepts of a line on the circles, we can show that

$$l\sqrt{\Sigma_1}+m\sqrt{\Sigma_2}+n\sqrt{\Sigma_3}=0$$

is the tangential equation of the conic.

If the line joining the centres of the circles is the major axis of the conic, we can easily show that the eccentricity (e) is given by the equation

$$\frac{e^2}{1-e^2}=\frac{lmn}{lr_1{}^2+mr_2{}^2+nr_3{}^2},$$

where r_1, r_2, r_3 are the radii of the circles. If the same line is the minor axis of the curve, we find

$$e^2 = \frac{lmn}{lr_1^2 + mr_2^2 + nr_3^2}.$$

It is to be observed that in the above l, m, n are taken so that

$$l + m + n = 0.$$

In the first case we see from Ex. 97 that the foci of the conic are the double points of the system in involution determined by the centres of similitude of the circles.

In the second case the foci are the points at which the centres of similitude of each pair of circles subtend a right angle.

143. Given two circles, show that there is a single parabola having double contact with them, and that the focus of the curve is the middle point of the centres of similitude of the circles.

144. Given four tangents to a conic, to find the locus of the centre of a circle of given radius having double contact with the curve.

Let α, β, γ, δ be the perpendiculars from a point on the four tangents, and let

$$l\alpha + m\beta + n\gamma + p\delta = 0 \qquad (1)$$

be an identical relation. Now if α', α are the perpendiculars from the pole of the chord of contact and centre of the circle, respectively, on a tangent of the curve, we have

$$\alpha'^2 = m^2 (\alpha^2 - r^2),$$

where r is the radius of the circle and m is a constant; hence from (1) we obtain

$$l\sqrt{(\alpha^2 - r^2)} + m\sqrt{(\beta^2 - r^2)} + n\sqrt{(\gamma^2 - r^2)} + p\sqrt{(\delta^2 - r^2)} = 0 \quad (2)$$

for the equation of the locus. This equation when cleared of radicals is found to be of the sixth degree, as terms of higher orders vanish identically. It can easily be shown that the locus passes through the circular points at infinity, and the points at infinity on the diagonals of the quadrilateral formed by the tangents. If we put $r = \delta$ in (2), we see that the locus of the foot of the normal at the point of contact of one of the tangents is

$$l\sqrt{(\alpha^2 - \delta^2)} + m\sqrt{(\beta^2 - \delta^2)} + n\sqrt{(\gamma^2 - \delta^2)} = 0,$$

which being divided by δ represents a cubic passing through the points where δ is met by α, β, γ.

145. By the same method as that employed in the preceding example we can show that, when we are given three tangents to a parabola, the locus of the centre of a circle of given radius, having double contact with the curve, is a curve of the fifth order. We can prove that this locus is unicursal as follows:—Since there is only one circle of given radius having double contact with a parabola, the co-ordinates of its centre must be capable of being expressed rationally in terms of the coefficients in the equation of the curve. But given three tangents to a parabola, the coefficients are quadratic functions of a parameter; therefore, &c.

146. Given four points on a conic, to find the locus of the centre of a circle of given radius having double contact with the curve.

Let α, β, γ, δ be the perpendiculars from the points on a line, then if l, m, n, p are the areas of the four triangles formed by the points, we have the identical relation

$$l\alpha + m\beta + n\gamma + p\delta = 0. \qquad (1)$$

Now if ρ_1, ρ_2, ρ_3, ρ_4 are the distances of a point from the four given points, and r is the given radius, we have

$$\rho_1^2 - r^2 = e^2\alpha^2, \quad \rho_2^2 - r^2 = e^2\beta^2, \text{ \&c.};$$

hence from (1) we obtain

$$l\sqrt{(\rho_1^2 - r^2)} + m\sqrt{(\rho_2^2 - r^2)} + n\sqrt{(\rho_3^2 - r^2)} + p\sqrt{(\rho_4^2 - r^2)} = 0,$$

which being cleared of radicals is found to represent a curve of the sixth order.

If the four points lie on a circle, we can show by the method employed in Salmon's *Conics*, Art. 288, Ex. 10, that the locus breaks up into two of the third degree.

If the four points are at the vertices of a parallelogram, we have $l = -m = n = -p$, and it can be easily shown, then, the locus reduces to a curve of the fourth order.

Putting $r = \rho_4$, we see that the normal at one of the points meets the arcs of the curve in points which lie on the conic

$$l\sqrt{(\rho_1^2 - \rho_4^2)} + m\sqrt{(\rho_2^2 - \rho_4^2)} + n\sqrt{(\rho_3^2 - \rho_4^2)} = 0.$$

When the given points lie on a circle this conic breaks up into two lines passing through the centre of the circle.

147. A circle of given radius has double contact with a conic inscribed in a fixed triangle; if the pole of the chord of contact lies on a fixed line, show that the locus of the centre of the circle is a curve of the fourth order.

VI.—CIRCLES CUTTING THE CURVE ORTHOGONALLY AT TWO POINTS.

148. There are two systems of circles which cut a conic orthogonally at two points, the lines joining the points being parallel to one of the axes of the curve.

If $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is the equation of the curve, the equation

$$x^2+y^2-2\frac{a^2}{x'}x+a^2+c^2-e^2x'^2=0 \qquad (1)$$

represents the circle cutting the curve orthogonally at the points where it is met by the line $x-x'=0$, and

$$x^2+y^2-2\frac{b^2}{y'}y+b^2-c^2+\frac{c^2}{b^2}y'^2=0 \qquad (2)$$

represents the circle cutting the curve orthogonally at points on the line $y-y'=0$.

If r is the radius and α the abscissa of the centre of the circle (1), we easily find

$$r^2=\frac{(\alpha^2-a^2)(\alpha^2-c^2)}{\alpha^2}, \qquad (3)$$

and if r' is the radius and β the ordinate of the centre of (2),

$$r'^2=\frac{(\beta^2-b^2)(\beta^2+c^2)}{\beta^2}. \qquad (4)$$

149. Since the equation (3) in the preceding example is unaltered if we interchange a and c, it follows that the circles

of the system (1) are also doubly orthogonal to the conic

$$\frac{x^2}{c^2} - \frac{y^2}{b^2} - 1 = 0.$$

The points on this conic will always be imaginary when those on the given curve are real, and real when the latter points are imaginary.

It may be observed that the circles of the system (2) are doubly orthogonal to the imaginary conic

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} + 1 = 0.$$

150. The envelope of the system of circles at Ex. 148, (1), is evidently

$$(x^2 + y^2 + a^2 + c^2)^3 - 27a^2c^2x^2 = 0,$$

which, being transformed to elliptic co-ordinates by means of the formulae

$$x^2 + y^2 + c^2 = \mu^2 + \nu^2, \quad cx = \mu\nu,$$

becomes $$(\mu^2 + \nu^2 + a^2)^3 - 27a^2\mu^2\nu^2 = 0\,;$$
but this is equivalent to

$$\mu^{\frac{2}{3}} + \nu^{\frac{2}{3}} + a^{\frac{2}{3}} = 0.$$

The envelope can also be written in this form, if the vertices of the curve are taken as the foci of the system of conics.

151. If pairs of circles having their centres on an axis, and cutting each other orthogonally, be described through a fixed point on a conic, the circles passing through the variable points where they meet the curve again have a common radical axis.

The circle having its centre on the axis of x and passing through the points x_1, x_2 on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

may be written

$$x^2 + y^2 - e^2(x_1 + x_2)x + e^2 x_1 x_2 - b^2 = 0. \qquad (1)$$

Now, if α is the abscissa of the fixed point, the circles

$$x^2 + y^2 - e^2(\alpha + x_1)x + e^2 \alpha x_1 - b^2 = 0,$$

$$x^2 + y^2 - e^2(\alpha + x_2)x + e^2 \alpha x_2 - b^2 = 0,$$

cut each other orthogonally if

$$e^4(\alpha + x_1)(\alpha + x_2) - 2e^2\alpha(x_1 + x_2) + 4b^2 = 0, \qquad (2)$$

subject to which condition the circle (1) passes through the fixed points determined by the equations

$$x = \frac{(2 - e^2)}{e^2}\alpha, \quad x^2 + y^2 = \left(1 + \frac{4}{e^2}\right)b^2 + e^2\alpha^2. \qquad (3)$$

These points satisfy the equation

$$x^2 + y^2 - 2\frac{a^2}{\alpha}x + a^2 + c^2 - e^2\alpha^2 = 0,$$

which represents the circle cutting the curve orthogonally at the points lying on $x = \alpha$. The locus of the points (3) is evidently a concentric conic; but if the curve is an equilateral hyperbola they lie on the axis of y.

152. If three circles of the system at Ex. 148, (1), be drawn through a point, show that the centroid of the three corresponding points on the curve lies on the axis of y.

153. Pairs of circles of the system at Ex. 148, (1), are described so as to have the same radius; show that their centres belong to a system in involution.

154. The circle cutting the parabola $y^2 - px = 0$ orthogonally at the points where it is met by the line $x - x' = 0$ may be written

$$x^2 + y^2 + 2x'x - 3x'^2 - px' = 0.$$

Its envelope is, therefore,

$$(2x - p)^2 + 12(x^2 + y^2) = 0,$$

which represents an imaginary conic.

155. Let S be the osculating circle at a point x', y' of the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Show that a parallel to the axis of y through the centre of S meets a circle concentric with the curve and cutting S orthogonally in the points of ultimate intersection of the circle

$$x^2 + y^2 - 2\frac{a^2}{x'}x + a^2 + c^2 - e^2x'^2 = 0.$$

156. Show that the length of the tangent from a focus to the circle (2), Ex. 148, is equal to the semi-diameter of the curve parallel to the tangent at y'.

Also show that the angle between the tangents from a focus to the circle is equal to $\pi - 2\phi$, where ϕ is the eccentric angle of y'.

157. If a circle of the system (1), Ex. 148, cut orthogonally a circle of the system (2) in the same example, we must have

$$a^2 + b^2 - c^2\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2}\right) = 0,$$

from which we can show that the line joining the centres of the circles touches the confocal conic

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2 - b^2}{a^2 + b^2}.$$

If a circle of one system cut orthogonally a circle of the other system belonging to a confocal conic, we can show that the line joining their centres touches a confocal conic.

158. Tangents are drawn from the focus c, o to circles

of the system (2), Ex. 148; show that their points of contact lie on the limaçon

$$(x^2 + y^2 - cx)^2 - b^2 (x^2 + y^2) = 0.$$

159. Lines are drawn through the focus c, o and the centres of circles of the system (2), Ex. 148; show that they meet the circles in points lying on the cubic

$$y^2 (c + x) - b^2 (c - x) = 0.$$

160. To describe through a point on the curve circles of the system (1), Ex. 148.

Eliminating y between the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

$$x^2 + y^2 - 2\frac{a^2}{x'} x + a^2 + c^2 - e^2 x'^2 = 0,$$

and dividing by $x - x'$, we obtain

$$e^2 (x^2 + x'x) - 2a^2 = 0;$$

hence two circles of the system may be described, and the circles passing through the variable points where they meet the curve again intersect the axis minor in two fixed points.

161. If two circles of the system (1), Ex. 148, be described to cut orthogonally a circle of the system (1), Ex. 89, show that the circle passing through the corresponding points on the curve meets the axis minor in the fixed points

$$y^2 = a^2 + c^2.$$

VII.—NORMALS.

162. A triangle is inscribed in the conic

$$S \equiv \frac{x^2}{a} + \frac{y^2}{b^2} - 1 = 0,$$

and circumscribed about the conic

$$S' \equiv \frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0,$$

to show that the normals to S, at the vertices of the triangle, pass through a point, and to find the locus of the point.

Writing down the conditions that the sides of the triangle

$$\frac{x}{a} \cos \tfrac{1}{2} (\alpha + \beta) + \frac{y}{b} \sin \tfrac{1}{2} (\alpha + \beta) - \cos \tfrac{1}{2} (\alpha - \beta) = 0, \text{ \&c.},$$

should touch S', and eliminating a' and b', we obtain

$$\sin (\alpha + \beta) + \sin (\beta + \gamma) + \sin (\gamma + \alpha) = 0, \qquad (1)$$

which shows that the normals always pass through a point. Now the hyperbola

$$S'' \equiv 2 (c^2 xy + b^2 y' x - a^2 x' y) = 0$$

passes through the feet of the normals drawn from x', y' to S; hence, since S'' circumscribes triangles circumscribed about S', we find from the invariant relation connecting the two latter conics

$$\frac{a^4 x'^2}{a'^2} + \frac{b^4 y'^2}{b'^2} - c^4 = 0. \qquad (2)$$

163. If in the preceding example

$$a' = \frac{a^3}{c^2}, \quad b' = \frac{b^3}{c^2}, \tag{1}$$

the locus (2) coincides with S. Thus we see that the normals to S at the extremities of chords touching the conic

$$\frac{x^2}{a^6} + \frac{y^2}{b^6} - \frac{1}{c^4} = 0 \tag{2}$$

intersect on the curve. We are permitted to assume the values (1) for a', b', as they are consistent with

$$\frac{a'}{a} - \frac{b'}{b} - 1 = 0,$$

the invariant relation connecting S and S'.

From Ex. 35 we see that the locus of the centre of the circumscribing circle is, in this case,

$$a^6x^2 + b^6y^2 = \tfrac{1}{4}a^4b^4.$$

Also the envelope of the circumscribing circle is

$$(x^2 + y^2 - a^2 - b^2)^2 - a^4b^4\left(\frac{x^2}{a^6} + \frac{y^2}{b^3}\right) = 0.$$

From Ex. 7 the locus of the centroid of the triangle is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{9}\frac{(a^2 + b^2)^2}{c^4}.$$

164. To find the area of the triangle considered in the preceding example.

Let x_1y_1, x_2y_2, x_3y_3 be the co-ordinates of the vertices of the triangle, then we have

$$2\Delta = \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ 1, & 1, & 1 \end{vmatrix};$$

therefore squaring we obtain

$$4\Delta^2 = \begin{vmatrix} \Sigma x^2, & \Sigma xy, & \Sigma x \\ \Sigma xy, & \Sigma y^2, & \Sigma y \\ \Sigma x, & \Sigma y, & 3 \end{vmatrix}. \qquad (1)$$

But eliminating y between

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \quad c^2 xy + b^2 y' x - a^2 x' y = 0,$$

we obtain

$$c^4 x^4 - 2a^2 c^2 x' x^3 + a^2 (a^2 x'^2 + b^2 y'^2 - c^4) x^2 + \&c. = 0\ ; \qquad (2)$$

hence $$x_1 + x_2 + x_3 + x' = \frac{2a^2}{c^2} x',$$

since x' is one of the roots of (2); therefore

$$\Sigma x = \frac{(a^2 + b^2)}{c^2} x',$$

and similarly $$\Sigma y = -\frac{(a^2 + b^2)}{c^2} y'.$$

We also find

$$\Sigma x^2 = \frac{2a^2}{c^4} \{(a^2 + 2c^2)x'^2 - b^2 y'^2 + c^4\},$$

$$\Sigma y^2 = \frac{2b^2}{c^4} \{(b^2 - 2c^2) y'^2 - a^2 x'^2 + c^4\},$$

$$\Sigma xy = -\frac{(a^2 + b^2)^2}{c^4} x' y'\ ;$$

hence, substituting these values in (1), and reducing by means of the equation $\dfrac{x'^2}{a^2} + \dfrac{y'^2}{b^2} - 1 = 0$, we get finally

$$\Delta^2 = \frac{a^2 b^2}{c^4} \left\{(a^2 - 2b^2)^3 \frac{y'^2}{b^4} - (2a^2 - b^2)^3 \frac{x'^2}{a^4}\right\}.$$

165. A chord of a conic is a tangent to a parabola which touches the axes of the curve; show that the normals at its extremities intersect on a curve of the third order.

166. The circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$$

passes through three points on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

at which the normals intersect in the same point; show that

$$\frac{a^2\alpha^2}{(k^2 + a^2)^2} + \frac{b^2\beta^2}{(k^2 + b^2)^2} = \frac{1}{4}.$$

167. From the point where a normal to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

touches the evolute two other normals are drawn to the curve; show that the line joining their feet is normal to the conic

$$a^2x^2 + b^2y^2 = \frac{a^4b^4}{c^4}.$$

168. If S and T are the invariants of the pencil which joins any point on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

to the feet of the four normals drawn to the curve from x, y, show that

$$\frac{S^3}{T^2} = \frac{(c^4 - a^2x^2 - b^2y^2)^3}{a^2b^2c^4x^2y^2}.$$

169. If from points on the line $lx + my + n = 0$ normals are drawn to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that the poles of the chords joining their extremities form a quadrilateral inscribed in the cubic

$$\frac{lx}{a^2}\left(1-\frac{y^2}{b^2}\right)+\frac{my}{b^2}\left(1-\frac{x^2}{a^2}\right)+\frac{n}{a^2-b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0.$$

170. A conic circumscribes a triangle so that the normals at the vertices pass through a point; to find the locus of the centre of the curve.

If θ is the angle which the chord joining the points α, β makes with the diameter bisecting it, it can be easily shown that $\cot\theta=\frac{c^2}{2ab}\sin(\alpha+\beta)$; hence we see that the condition (*Conics*, Art. 231, Ex. 10) that the three normals should meet in a point can be written in the form

$$\cot\theta_1+\cot\theta_2+\cot\theta_3=0, \qquad (1)$$

where θ_1, θ_2, θ_3 are the angles which the sides of the triangle make with the diameters bisecting them. But if α, β, γ are the perpendiculars from a point on the sides of the triangle, we have

$$\cot\theta_1=\frac{\beta\sin B-\gamma\sin C+\alpha\sin(B-C)}{2\alpha\sin B\sin C},$$

and similar values for θ_2, θ_3; hence, from (1), we obtain

$$\frac{\alpha}{\sin A}(\beta^2-\gamma^2)+\frac{\beta}{\sin B}(\gamma^2-\alpha^2)+\frac{\gamma}{\sin C}(\alpha^2-\beta^2)=0, \quad (2)$$

which represents a cubic passing through the vertices of the triangle, the centroid, and the centres of the circles touching the sides.

It may be observed that if conics be inscribed in the triangle, so that the axis major passes through the centroid, the foci will lie on the cubic (2).

171. In the preceding example show that the locus of the points through which the normals pass is the cubic

$$(\cos A - \cos B \cos C)\,\alpha\,(\beta^2 - \gamma^2) + (\cos B - \cos A \cos C)\,\beta\,(\gamma^2 - \alpha^2) + (\cos C - \cos A \cos B)\,\gamma\,(\alpha^2 - \beta^2) = 0.$$

172. To find the condition that the normals at six points on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ should be all touched by the same conic.

Expressing that the normal whose equation is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} - c^2 = 0$$

touches a conic given by the general equation, we obtain an equation of the eighth degree in $\tan\frac{1}{2}\phi$, between the roots of which we find three relations by eliminating the constants in the equation of the conic. Eliminating from these relations two of the roots, we have the condition required

$$PQ - RS = 0,$$

where $P = \Sigma \cos\frac{1}{2}(\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6)$,

$Q = \Sigma \sin(\phi_1 + \phi_2 + \phi_3 + \phi_4)$,

$R = \Sigma \cos(s - \phi_1)$, $\quad S = \sin 2s + \Sigma \sin(\phi_1 + \phi_2)$,

$2s$ being equal to $\Sigma\phi$.

If the normals at six points on the parabola $y^2 - px = 0$ are all touched by the same conic, we find

$$\Sigma y = 0.$$

173. Three normals are drawn from a point P of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

to the curve; show that the product of their lengths is equal to

$$\frac{2a^2 b^2}{a^2 - b^2}\rho,$$

where ρ is the radius of curvature at P.

174. Normals are drawn from a point x, y to the parabola $y^2 - 4mx = 0$; if Δ is the area of the triangle formed by their feet, show that

$$\Delta^2 = 4m(x - 2m)^3 - 27m^2y^2.$$

175. Normals to the parabola $y^2 - 4mx = 0$ include a constant angle; to find the locus of their intersection.

Let us write the equation of the normal in the form

$$y - tx + \tfrac{1}{2}pt + \tfrac{1}{4}pt^3 = 0, \qquad (1)$$

where t is the tangent of the angle the normal makes with the axis of x; then if α, β, γ are the roots of the equation (1) in t, we form the equation whose roots are $\frac{(\alpha - \beta)^2}{(1 + \alpha\beta)^2}$, &c. We thus find for the locus

$$t^2\{2m^2 - x^2 + mx + t^2(y^2 + 3m^2 - mx)\}^2$$
$$- m(1 + t^2)^2\{4(x - 2m)^3 - 27my^2\} = 0,$$

where t is the tangent of the given angle.

176. Triangles are inscribed in the parabola

$$V \equiv y^2 - px = 0,$$

and circumscribed about the parabola

$$(\alpha x + \beta y)^2 - 4p\beta^2 x + \gamma\alpha y - p\beta\gamma = 0;$$

show that the normals to V at the vertices of the triangle pass through a point, and show that the locus of this point is a right line.

177. To draw a normal to an equilateral hyperbola from a point on the curve.

The curve referred to the asymptotes being written in the form

$$2xy - a^2 = 0\,; \qquad (1)$$

the hyperbola which passes through the feet of the normals from x', y' to the curve is

$$x^2 - y^2 - x'x + y'y = 0. \qquad (2)$$

Solving, then, for x from (1) and (2), and remembering that $2x'y' = a^2$, we find

$$x^3 = -\tfrac{1}{2}\,a^2y',$$

and, similarly,

$$y^3 = -\tfrac{1}{2}\,a^2x'.$$

Thus we see that but one real normal can be drawn.

VIII.—LINES MAKING A CONSTANT ANGLE WITH THE CURVE.

178. To find the locus of the poles of lines making a constant angle with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

If ϕ is the eccentric angle at a point on the curve, and m the tangent of the given angle, the equation of the line may be written

$$x (b \cos \phi + ma \sin \phi) + y (a \sin \phi - mb \cos \phi) - (ab + mc^2 \sin \phi \cos \phi) = 0. \quad (1)$$

Comparing this equation with

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 = 0,$$

we get

$$\frac{x'}{a^2} = \frac{b \cos \phi + ma \sin \phi}{ab + mc^2 \sin \phi \cos \phi},$$

$$\frac{y'}{b^2} = \frac{a \sin \phi - mb \cos \phi}{ab + mc^2 \sin \phi \cos \phi};$$

hence, eliminating ϕ, and omitting the accents, the equation of the locus is

$$m^2 \left(\frac{c^4 x^2 y^2}{a^6 b^6} - \frac{x^2}{a^6} - \frac{y^2}{b^6}\right) - \frac{2mc^2 xy}{a^4 b^4} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \frac{1}{a^2 b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^4}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0,$$

which represents a quartic having a node at the origin and two nodes at infinity. We see thus that the envelope of the line (1) is a curve of the sixth order with six cusps.

179. To find the condition that three lines making an angle $\tan^{-1} m$ with

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

at the points α, β, γ should meet in a point.

Since the equation of the line making the angle $\tan^{-1} m$ with the curve at the point x', y' is

$$a^2 b^2 \left(\frac{xx'}{a^2} + \frac{yy'}{b^2} - 1 \right) + m\,(a^2 y' x - b^2 x' y - c^2 x' y') = 0,$$

it follows that (see *Conics*, Art. 181, Ex. 1) the hyperbola

$$S' \equiv m\,(c^2 xy + b^2 y' x - a^2 x' y) - b^2 x' x - a^2 y' y + a^2 b^2 = 0 \quad (1)$$

passes through the points on S, at which lines making the angle $\tan^{-1} m$ with the curve intersect in x', y'; hence, if P, Q, are a pair of lines passing through the latter points, we must have

$$S' + kS \equiv \lambda PQ, \qquad (2)$$

where, in terms of the eccentric angles,

$$P = \frac{x}{a} \cos \tfrac{1}{2}(\alpha + \beta) + \frac{y}{b} \sin \tfrac{1}{2}(\alpha + \beta) - \cos \tfrac{1}{2}(\alpha - \beta),$$

$$Q = \frac{x}{a} \cos \tfrac{1}{2}(\gamma + \delta) + \frac{y}{b} \sin \tfrac{1}{2}(\gamma + \delta) - \cos \tfrac{1}{2}(\gamma - \delta).$$

Equating, then, the co-efficients of x^2, y^2, xy, and the absolute terms in this identity, we obtain

$$k = \lambda \cos \tfrac{1}{2}(\alpha + \beta) \cos \tfrac{1}{2}(\gamma + \delta) = \lambda \sin \tfrac{1}{2}(\alpha + \beta) \sin \tfrac{1}{2}(\gamma + \delta),$$

$$mc^2 ab = \lambda \sin \tfrac{1}{2}(\alpha + \beta + \gamma + \delta),$$

$$a^2 b^2 - k = \lambda \cos \tfrac{1}{2}(\alpha - \beta) \cos \tfrac{1}{2}(\gamma - \delta);$$

hence $$\alpha + \beta + \gamma + \delta = \pi, \quad (3)$$

and $$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = \frac{2ab}{mc^2}. \quad (4)$$

180. The envelope of the line (1), Ex. 178, may be most easily obtained by expressing that the conics S and S' in the preceding example touch one another (*Conics*, Art. 372). We see thus that the envelope is of the sixth order. We find the four points on the curve corresponding to the cusps of the envelope by putting $\alpha = \beta = \gamma$ in (4), Ex. 179, when we get

$$\sin 2\alpha = \frac{2ab}{3mc^2}.$$

The two remaining cusps are at infinity.

181. If through any point x', y' of the hyperbola

$$(y + mx)(x - my) - mc^2 = 0$$

lines be drawn to make the angle $\tan^{-1} m$ with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that their feet lie on the lines

$$mc^2x - a^2(y' + mx') = 0,$$
$$mc^2y - b^2(x' - my') = 0.$$

182. The circle passing through x', y' and the points of contact of the tangents from x', y' to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

meets the curve again on the line L. Show that the centre of the hyperbola (1), Ex. 179, lies on L.

183. If triangles be inscribed in the conic

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and circumscribed about the concentric conic, whose tangential equation is

$$\Sigma \equiv A\lambda^2 + B\mu^2 + C\nu^2 + 2H\lambda\mu = 0,$$

we have from (3) and (4), Ex. 39,

$$\left(\frac{A}{a^2} + \frac{B}{b^2}\right)\cos\phi + \frac{2H}{ab}\sin\phi = Cp,$$

$$\left(\frac{A}{a^2} + \frac{B}{b^2}\right)\sin\phi - \frac{2H}{ab}\cos\phi = Cq\,;$$

hence $$p\sin\phi - q\cos\phi = \frac{2H}{abC};$$

but $p\sin\phi - q\cos\phi = \sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)$;

thus we see at once from (4), Ex. 179, that lines making the constant angle $\tan^{-1}\left(\frac{a^2b^2C}{c^2H}\right)$ with S at the vertices of the triangle pass through a point.

To find the locus of this point we express the invariant condition that the hyperbola (1), Ex. 179, should circumscribe triangles circumscribed about Σ; we thus find

$$a'a^4(y+mx)^2 + b'b^4(x-my)^2 - 2h'a^2b^2(y+mx)(x-my) - m^2c^4 = 0,$$

where $$a'x^2 + b'y^2 + 2h'xy - 1 = 0$$

is the equation of Σ in x, y co-ordinates.

184. Lines making a constant angle with a conic at the vertices of an inscribed triangle pass through a point on the curve; show that the locus of the centroid of the triangle is a concentric conic.

185. From the point of intersection of two lines making a constant angle $\tan^{-1}m$ with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

at the extremities of a chord which passes through the fixed point x', y', the two other lines are drawn which make the same angle with the curve; show that the chord joining the feet of the latter pair of lines touches the parabola whose tangential equation is

$$a^2b^2(a^2\lambda^2 + b^2\mu^2) + mc^2(b^2x'\mu\nu + a^2y'\nu\lambda - a^2b^2\lambda\mu) = 0.$$

186. The locus of the intersection of lines making a constant angle with a conic at the extremities of a chord which passes through a fixed point is a curve of the third order (see *Conics*, Art. 370, Ex.). If the fixed point is on the diameter which cuts the curve at the given angle, the locus reduces to a conic, as the diameter in this case is part of the locus. The locus also reduces to a conic if the fixed point is at infinity.

187. From the point where a line making a constant angle with the conic touches its envelope the two other lines are drawn which make the same angle with the curve; show that the line joining their feet cuts a concentric conic at a constant angle.

188. A conic circumscribes a fixed triangle, so that lines making a given angle θ with the curve at the vertices pass through a point; show that the locus of its centre referred to the triangle is the cubic

$$\frac{\alpha}{\sin A}(\beta^2 - \gamma^2) + \frac{\beta}{\sin B}(\gamma^2 - \alpha^2) + \frac{\gamma}{\sin C}(\alpha^2 - \beta^2) + 2\cot\theta\,\alpha\beta\gamma = 0$$

(*see* Ex. 170).

189. If α, β, γ, δ are constants, and x, y rectangular co-ordinates, show that the line

$$\alpha x\cos\phi + \beta y\sin\phi = \gamma + \delta\sin\phi\cos\phi$$

cuts at a constant angle a conic having the origin for centre. Also show that this cannot be the case if $\alpha = \beta$.

190. If three lines making the given angle $\tan^{-1} m$ with the parabola $y^2 - 4ax = 0$ at the points y_1, y_2, y_3 pass through a point, we find

$$y_1 + y_2 + y_3 = \frac{2a}{m}. \qquad (1)$$

Now $$y_1 + y_2 + y_3 + y_4 = 0$$

is the condition that four points on the curve should lie on a circle; hence we see from (1), that the circle passing through the three points meets the curve again in the fixed point

$$y = -\frac{2a}{m}.$$

191. Triangles are inscribed in a parabola S, and circumscribed about a parabola S'; show that lines making a certain constant angle with S at the vertices of one of the triangles pass through a point, and show that the locus of this point is a right line.

192. If two lines are drawn through the point y' on the parabola $y^2 - 4ax = 0$, to meet the curve again at the angle $\tan^{-1} m$, we find that the equation of the line joining their feet is

$$4max + (my' - 2a)\, y + 2a\,(y' + 4ma) = 0,$$

which, when y' varies, passes through the fixed point

$$y = -\frac{2a}{m}, \quad x = -\frac{a}{m^2}(1 + 2m^2).$$

The locus of this point for different values of m is the equal parabola

$$y^2 + 4a\,(x + 2a) = 0.$$

IX.—OSCULATING CIRCLES.

193. A triangle is circumscribed about a conic Σ, and inscribed in a confocal conic Σ'; to show that the osculating circles at the points of contact of the sides are all touched by the fourth common tangent of Σ and one of the circles touching the sides.

If Σ and Σ' are both ellipses, it is evident that the tangents to Σ', at the vertices of the triangle, are the external bisectors of the angles; hence its equation in trilinear co-ordinates must be

$$\alpha\beta + \beta\gamma + \gamma\alpha = 0,$$

and in tangential co-ordinates

$$\Sigma' \equiv \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu - 2\nu\lambda - 2\lambda\mu = 0. \qquad (1)$$

But Σ being confocal with Σ' must be of the form $\Sigma' + k\Omega$, where

$$\Omega \equiv \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos A - 2\nu\lambda\cos B - 2\lambda\mu\cos C = 0 \qquad (2)$$

represents the circular points at infinity; hence, since Σ must not contain the terms λ^2, μ^2, ν^2, we have

$$\Sigma = \lambda\mu\sin^2\tfrac{1}{2}A + \nu\lambda\sin^2\tfrac{1}{2}B + \lambda\mu\sin^2\tfrac{1}{2}C. \qquad (3)$$

Now $$\Sigma - (\lambda\sin^2\tfrac{1}{2}B + \mu\sin^2\tfrac{1}{2}A)(l\lambda + m\mu) = 0 \qquad (4)$$

represents a conic having contact of the second order with Σ on the side opposite the vertex ν; and if we express that this conic (4) passes through the points represented by $\Omega = 0$, we should obtain the equation of one of the osculating circles.

We thus obtain the equation

$$\Omega - \left\{\lambda\left(\frac{2\sin^2\frac{1}{2}B}{\sin^2\frac{1}{2}C} - 1\right) + \mu\left(\frac{2\sin^2\frac{1}{2}A}{\sin^2\frac{1}{2}C} - 1\right) + \nu\right\}^2 = 0. \quad (5)$$

But this is evidently satisfied by

$$\lambda = \frac{1}{\cos B - \cos C}, \quad \mu = \frac{1}{\cos C - \cos A}, \quad \nu = \frac{1}{\cos A - \cos B};$$

and these are evidently the co-ordinates of the fourth common tangent of Σ and the circle inscribed in the triangle. The three osculating circles have, therefore, this line for a common tangent.

If Σ and Σ' are not both ellipses, the osculating circles are touched by the fourth common tangent of Σ and one of the exscribed circles.

The above result can be readily proved by means of elliptic functions; for, by Chasles's theorem, the extremities of the diagonals of the quadrilateral formed by the common tangents of a conic and a circle lie on a confocal conic; hence, from Ex. 44, we see that if four tangents are touched by the same circle we must have

$$u_1 + u_2 + u_3 + u_4 = 0, \quad \text{or} \quad 4mK.$$

Now three tangents coincide at the point of contact of an osculating circle; hence, for the points of contact u_1, u_2, u_3 of three osculating circles which touch the tangent u, we have

$$3u_1 + u = 0,$$

whence $u_1 = -\frac{1}{3}u$, and $u_2 = -\frac{1}{3}u + \frac{4}{3}K$, $u_3 = -\frac{1}{3}u + \frac{8}{3}K$,

from which it follows that the tangents u_1, u_2, u_3 are touched by a circle touching u. There are nine osculating circles touching a given tangent, but six of these are imaginary, corresponding to the imaginary periods of u.

194. Show that but one real osculating circle, besides the one at the point of contact, can be described to touch a given tangent to a parabola.

195. Six osculating circles of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

are described to cut orthogonally the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0\ ;$$

to show that their centres lie on a conic.

Expressing that the osculating circle whose centre is x, y and radius r cuts S orthogonally, we obtain

$$2gx + 2fy + c + x^2 + y^2 - r^2 = 0. \qquad (1)$$

But from the equation of the osculating circle (*Conics*, Art. 251, Ex. 3) we have, in terms of the eccentric angle,

$$x = \frac{c^2}{a}\cos^3\theta, \quad y = -\frac{c^2}{b}\sin^3\theta,$$

$$x^2 + y^2 - r^2 = (a^2 - 2b^2)\cos^2\theta + (b^2 - 2a^2)\sin^2\theta\ ; \quad (2)$$

therefore $\qquad a^2x^2 + b^2y^2 = c^4(1 - 3\sin^2\theta\cos^2\theta)\ ;$

hence from (1) and (2) we obtain

$$(2gx + 2fy + c)^2 - (a^2 + b^2)(2gx + 2fy + c) - 3(a^2x^2 + b^2y^2)$$
$$+ a^4 + b^4 - a^2b^2 = 0, \qquad (3)$$

which proves the theorem.

If the curve is an equilateral hyperbola, the conic (3) has double contact with the curve.

196. If four osculating circles of the parabola $y^2 - px = 0$

are cut orthogonally by the same circle, show that the ordinates of the points of contact are connected by the relation

$$\Sigma \frac{1}{y} = 0.$$

197. Three circles osculate the parabola $y^2 - px = 0$ at the points y_1, y_2, y_3; show that the equation of the circle cutting them orthogonally is

$$p^2(x^2+y^2) - \frac{1}{p}(p_2^2 - p_1 p_3)\,x - \tfrac{3}{4}(p_1 p_2 - p_3)\,y$$
$$+ \tfrac{1}{2}(p_2^2 - p_1 p_3) + \frac{3p_3^2}{p^2} = 0,$$

where $\quad p_1 = \Sigma y_1, \quad p_2 = \Sigma y_1 y_2, \quad p_3 = y_1 y_2 y_3.$

198. If the osculating circle of a conic cut the director circle at an angle θ, show that $\cos\theta = \frac{3p}{2k}$, where p is the perpendicular from the centre on the tangent at the point of contact, and k is the radius of the director circle.

199. Let the tangent to a hyperbola at a point P meet the asymptotes in A, B; if perpendiculars to the asymptotes at A, B meet the normal at P in A', B', show that the middle point of A', B' is the centre of curvature at P.

200. The tangent to a conic S at the point P meets a concentric, similar and similarly situated conic S' in A, B; if the normals to S' at A, B meet the normal to S at P in A', B', show that the middle point of A', B' is the centre of curvature at P.

201. If S is the square of the tangent drawn from the point whose eccentric angle is θ on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

to the circle passing through the points α, β, γ on the curve, we have, from Ex. 1,

$$S = a^2\cos^2\theta + b^2\sin^2\theta - \tfrac{1}{2}c^2\cos\theta\,\{\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma)\}$$

$$+ \tfrac{1}{2}c^2\sin\theta\,\{\sin\alpha + \sin\beta + \sin\gamma + \sin(\alpha+\beta+\gamma)\}$$

$$+ \tfrac{1}{2}c^2\,\Sigma\cos(\alpha+\beta) - \tfrac{1}{2}(a^2+b^2)$$

$$= \tfrac{1}{2}c^2\{\cos 2\theta - \cos(\alpha+\beta+\gamma-\theta) + \Sigma\cos(\alpha+\beta) - \Sigma\cos(\theta+\gamma)\}$$

$$= c^2\sin\tfrac{1}{2}(\theta+\alpha+\beta+\gamma)\{\sin\tfrac{1}{2}(\alpha+\beta+\gamma-3\theta)$$

$$+ \sin\tfrac{1}{2}(\theta+\alpha-\beta-\gamma) + \sin\tfrac{1}{2}(\theta+\beta-\gamma-\alpha) + \sin\tfrac{1}{2}(\theta+\gamma-\alpha-\beta)\}$$

$$= 4c^2\sin\tfrac{1}{2}(\theta+\alpha+\beta+\gamma)\sin\tfrac{1}{2}(\theta-\alpha)\sin\tfrac{1}{2}(\theta-\beta)\sin\tfrac{1}{2}(\theta-\gamma). \quad (1)$$

Putting, then, $\alpha = \beta = \gamma$ in this expression (1) for an osculating circle, we get

$$S = 4c^2\sin\tfrac{1}{2}(\theta + 3\alpha)\sin^3\tfrac{1}{2}(\theta - \alpha). \qquad (2)$$

Now if we have three osculating circles passing through the same point on the curve, the eccentric angles of their points of contact are connected by the relations (*Conics*, Art. 244, Ex. 3)

$$\alpha - \beta = \tfrac{2}{3}\pi, \quad \alpha - \gamma = \tfrac{4}{3}\pi;$$

and then $\sin\tfrac{1}{2}(\theta-\alpha) + \sin\tfrac{1}{2}(\theta-\beta) + \sin\tfrac{1}{2}(\theta-\gamma) = 0. \quad (3)$

Thus we see from (2) and (3) that if the curve be referred to three such osculating circles S_1, S_2, S_3, its equation will be

$$\sqrt[3]{S_1} + \sqrt[3]{S_2} + \sqrt[3]{S_3} = 0. \qquad (4)$$

If we substitute for S_1, S_2, S_3 expressions in Cartesian co-ordinates, and clear equation (4) of radicals, terms of higher orders than the fourth will be found to vanish identically. The result is then divisible by the square of the distance from the point common to S_1, S_2, S_3, and the remaining factor gives the equation of the curve.

202. Three osculating circles of a conic pass through the same point on the curve; if ϕ, χ, ψ are the angles at which they intersect, and A, B, C the angles of the triangle formed by their points of contact, show that

$$\phi = 3A - \pi, \quad \psi = 3B - \pi, \quad \chi = 3C - \pi.$$

203. Three osculating circles of a conic pass through the same point on the curve; show that the triangle formed by their centres is similar to the triangle formed by the extremities of the diameters conjugate to those drawn to the points of contact.

Also show that the ratio of corresponding sides is equal to $\frac{3}{4}\left(\frac{a^2 - b^2}{ab}\right)$, where a, b are the semi-axes of the curve.

X.—CONICS HAVING DOUBLE CONTACT WITH A FIXED CONIC.

204. A conic has double contact with a fixed conic and a fixed circle; to find the locus of the foci.

If the tangential equation of the fixed conic is

$$\Sigma \equiv a^2\lambda^2 + b^2\mu^2 - 1 = 0,$$

and that of the circle

$$\Sigma' \equiv (x'\lambda + y'\mu - 1)^2 - r^2(\lambda^2 + \mu^2),$$

we have seen in Ex. 44 that, when $\Sigma + \frac{h^2}{r^2}\Sigma'$ breaks up into factors, h^2 is a root of the cubic equation

$$\frac{x'^2}{a^2 - h^2} + \frac{y'^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0; \qquad (1)$$

hence if we write $\quad \Sigma + \frac{h^2}{r^2}\Sigma' = EF, \qquad (2)$

where E and F are the extremities of one of the diagonals of the quadrilateral formed by the common tangents of Σ and Σ', the equation

$$\theta^2 E^2 + 2\theta\left(\Sigma - \frac{h^2}{r^2}\Sigma'\right) + F^2 = 0 \qquad (3)$$

(see *Conics*, Art. 287) represents a conic having double contact with Σ and Σ'. Writing this equation (3) in the form

$$(\theta E + F)^2 - 4\theta\frac{h^2}{r^2}(\alpha\lambda + \beta\mu - 1)^2 + 4\theta h^2(\lambda^2 + \mu^2) = 0,$$

we see that the points

$$\theta E + F \pm \frac{2h}{r}\sqrt{\theta}\,(\alpha\lambda + \beta\mu - 1) = 0 \qquad (4)$$

are foci. Taking, then, the envelope of this equation (4), we have, for the locus of the foci,

$$EF - \frac{h^2}{r^2}(\alpha\lambda + \beta\mu - 1)^2 = 0,$$

or from (2) $$\Sigma - h^2(\lambda^2 + \mu^2) = 0, \qquad (5)$$

which represents the confocal conic passing through the points E and F (see Ex. 44 (2)).

This conic gives the locus of the foci, when the major axis of the variable conic passes through the centre of Σ'; for the points represented by the equation (4) are evidently collinear with $x'\lambda + y'\mu - 1 = 0$, which represents the centre of Σ'. When the minor axis of the variable conic passes through the centre Σ', the foci are the anti-points (Salmon's *Higher Plane Curves*, Art. 139) of two points on the conic (5) which are collinear with the fixed point x', y'. To find the locus in this case, let

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 \equiv S = 0$$

be the equation of (5) in x, y co-ordinates; forming, then, the equation of the chords of intersection of S and

$$(x - x')^2 + (y - y')^2 = 0$$

(see *Conics*, Art. 370, Ex.), and expressing that this equation is satisfied by the co-ordinates of the fixed point, we obtain the locus required

$$\{(x-x')^2+(y-y')^2\}^2+\left(\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}-1\right)(x^2+y^2-a'^2-b'^2)\{(x-x')^2+(y-y')^2\}$$
$$-a'^2b'^2\left(\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}-1\right)^2\left(\frac{x^2}{a'^2}+\frac{y^2}{b'^2}-1\right)=0.$$

Writing this bicircular quartic in the form

$$\{(x-x')^2+(y-y')^2+\tfrac{1}{2}S'(x^2+y^2-a'^2-b'^2)\}^2$$
$$=\tfrac{1}{4}S'^2\{(y^2+(x+c)^2\}\{y^2+(x+c)^2\},$$

where $$S'\equiv\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}-1,$$

we see that it has two foci in common with Σ. We can also show that the quartic has a node at the foot of the perpendicular from x', y' on its polar with regard to S.

Hence the complete locus for the three systems of variable conics consists of three confocal conics and three nodal bicircular quartics.

205. A conic has double contact with a fixed circle and passes through two fixed points; show that the locus of the foci consists of two confocal conics and two nodal bicircular quartics.

206. A conic has double contact with a fixed circle and touches two fixed lines; show that the locus of the foci consists of two pairs of lines and two circles.

207. A conic has double contact with a fixed circle, passes through a fixed point, and touches a fixed line; show that the locus of the foci consists of two conics and two nodal bicircular quartics.

208. A conic has double contact with a fixed circle and touches two fixed conics having double contact with the circle; to find the locus of the foci.

Let $x^2+y^2-k^2\equiv J=0$ be the x, y equation of the fixed circle, and $\nu^2-k^2(\lambda^2+\mu^2)=0$ its tangential equation; then

$$m^2\{\nu^2-k^2(\lambda^2+\mu^2)\}-(a\lambda+\beta\mu+\nu)^2=0 \qquad (1)$$

represents a conic having double contact with J. But the pair of foci of this conic (1) which are collinear with the centre of J are evidently determined by the equation

$$a\lambda + \beta\mu + \nu \pm m\nu = 0\,;$$

hence we may write (1)

$$m^2\{\nu^2 - k^2(\lambda^2 - \mu^2)\} - \{(m-1)(x\lambda + y\mu) - \nu\}^2 = 0, \quad (2)$$

where x, y are the co-ordinates of one of the foci. Now, if this conic (2) touch the conic

$$m_1^2\{\nu^2 - k^2(\lambda^2 + \mu^2)\} - \{(m_1 - 1)(x_1\lambda + y_1\mu) - \nu\}^2 = 0,$$

we must have (*Conics*, Art. 387)

$$\{(m-1)^2(x^2+y^2) - k^2(1+m^2)\}\{(m_1-1)^2(x_1^2+y_1^2) - k^2(1+m_1^2)\}$$
$$- \{(m-1)(m_1-1)(xx_1+yy_1) - k^2(1 \pm mm_1\}^2 = 0;$$

but this relation can be written in the form

$$\left\{x^2 + y^2 + k^2\left(\frac{m+1}{m-1}\right)\right\}^{\frac{1}{2}} + \left\{x_1^2 + y_1^2 + k^2\left(\frac{m_1+1}{m_1-1}\right)\right\}^{\frac{1}{2}}$$
$$= \{(x-x_1)^2 + (y-y_1)^2\}^{\frac{1}{2}} = \rho_1 \quad \text{say, } (3)\,;$$

hence, if the conic (2) touch two fixed conics of the system, we have, from (3),

$$(\rho_1 \pm \rho_2)^2 = (\sqrt{S_1} \pm \sqrt{S_2})^2, \qquad (4)$$

where $\quad S_1 = x_1^2 + y_1^2 + k^2\left(\dfrac{m_1+1}{m_1-1}\right), \quad S_2 = \&c.$

But this equation (4) represents a pair of confocal conics, of which each focus is also that of one of the fixed conics. Taking account, therefore, of all the foci of the fixed conics, we see that the locus consists of four pairs of conics whose foci are formed out of the foci of the fixed conics. These four pairs of conics give the locus of the foci when the major axis of the variable conic passes through the centre of J. When the minor axis of the variable conic passes through the centre of J, the foci are the anti-points of the imaginary foci which are collinear with the centre of J. If $x_1 y_1$, $x_2 y_2$ are the co-ordinates of the latter pair of foci, and x, y those of a real focus, we have

$$2x = x_1 + x_2 \pm \sqrt{-1}\,(y_1 - y_2), \quad 2y = y_1 + y_2 \mp \sqrt{-1}\,(x_1 - x_2). \quad (5)$$

But if $x_1\,y_1$ is a focus of the conic (2), it is easily shown that the other focus is

$$\frac{1-m}{1+m}x_1, \quad \frac{1-m}{1+m}y_1,$$

and if
$$(x-\alpha)^2 + (y-\beta)^2 = (ax + by + c)^2 \quad (6)$$

is the equation of the locus of x_1, y_1, we have, from (3),

$$x_1^2 + y_1^2 + k^2\frac{(m+1)}{m-1} = (ax_1 + by_1 + c - \sqrt{S})^2,$$

where
$$S = \alpha^2 + \beta^2 + k^2\frac{(m'+1)}{m'-1};$$

hence $\dfrac{m+1}{m-1}$ is equal to an expression of the form

$$a'x_1 + b'y_1 + c'.$$

We have then, from (5),

$$2x = x_1\left(1 + \frac{1}{\gamma}\right) \pm \sqrt{-1}\,y_1\left(1 - \frac{1}{\gamma}\right),$$

$$2y = y_1\left(1 + \frac{1}{\gamma}\right) \mp \sqrt{-1}\,x_1\left(1 - \frac{1}{\gamma}\right),$$

where
$$\gamma = a'x_1 + b'y_1 + c',$$

and x_1, y_1 are connected by the relation

$$(x_1 - \alpha)^2 + (y_1 - \beta)^2 = (ax_1 + by_1 + c)^2.$$

We thus find, by eliminating x_1, y_1 from these equations, that the locus of x, y is a bicircular quartic. There are eight such quartics altogether corresponding to the different conics. The entire locus thus consists of eight conics and eight bicircular quartics.

209. A conic has double contact with a fixed conic, and touches two fixed parallel lines; show that the envelope of the asymptotes consists of two conics—concentric, similar, and similarly situated with the fixed conic.

210. A conic has double contact with two fixed confocal conics; to find the locus of its foci.

Let

$$\Sigma \equiv a^2\lambda^2 + b^2\mu^2 - \nu^2 = 0,$$

$$\Sigma' \equiv a'^2\lambda^2 + b'^2\mu^2 - \nu^2 = 0$$

be the tangential equations of the fixed conics, and let

$$a'^2 - a^2 = b'^2 - b^2 = h^2, \quad a^2 - b^2 = c^2.$$

Then $\theta^2 h^2 (c\lambda + \nu)^2 + 2\theta (b'^2 \Sigma + b^2 \Sigma') + h^2 (c\lambda - \nu)^2 = 0$ (1)

represents a conic having double contact with Σ and Σ'. The axis of x is one of the principal axes of the conics of this system.

To find the foci of (1), we have (*Conics*, Art. 258)

$$C(x^2 - y^2) - 2Gx + A - B = 0, \quad (2)$$

$$y(Cx - G) = 0, \quad (3)$$

where

$$C = h^2(\theta^2 + 1) - 2(b^2 + b'^2)\theta,$$

$$G = ch^2(\theta^2 - 1), \quad A - B = c^2h^2(\theta^2 + 1) + 2c^2(b^2 + b'^2)\theta;$$

hence, omitting the factor $y = 0$, and eliminating θ between the equations (2) and $Cx - G = 0$, we get

$$x^2 + y^2 \pm c\sqrt{-1}\,\frac{(b^2 + b'^2)}{bb'}\,y - c^2 = 0,$$

which represents a pair of circles passing through the foci and the intersection of Σ and Σ'.

For the system of conics, one of whose principal axes coincides with the axis of y, the locus is found to be

$$x^2 + y^2 \pm c\,\frac{(a^2 + a'^2)}{aa'}\,x + c^2 = 0.$$

The equation

$$h^2 \cos\theta\,(\lambda^2 - \mu^2) + 2h^2 \sin\theta\,\lambda\mu + \Sigma + \Sigma' = 0 \qquad (4)$$

represents the system of conics having double contact with Σ and Σ', which is concentric with them. For this system the equations for determining the foci are

$$x^2 - y^2 - c^2 - h^2\cos\theta = 0, \quad 2xy - h^2\sin\theta = 0\,;$$

hence the locus is

$$(x^2 + y^2)^2 - 2c^2(x^2 - y^2) + c^4 - h^4 = 0,$$

which represents a confocal oval of Cassini passing through the intersection of the given conics.

211. The director circle of the conics of the system (4) in the preceding example is

$$x^2 + y^2 - (a^2 + b^2 + h^2) = 0,$$

which is absolutely fixed. This circle is the locus of points through which tangents to Σ cut at right angles tangents to Σ'.

Putting $a^2 + b^2 + h^2 = 0$, we see that an infinite number of equilateral hyperbolæ can be described to have double contact with the confocal hyperbolæ

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad \frac{x^2}{b^2} - \frac{y^2}{a^2} = 1.$$

212. The differential equation in elliptic co-ordinates

$$\frac{d\mu}{\sqrt{\{(a^2-\mu^2)(\mu^2-a'^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2-\nu^2)(\nu^2-a'^2)\}}} = 0 \quad (1)$$

represents a system of conics having double contact with the confocal conics $\mu = a$, $\nu = a'$; for, by the theory of elliptic functions, the integral of this equation (1) can be written in either of the forms

$$\left.\begin{aligned} A\mu\nu + B\sqrt{\{(a^2-\mu^2)(a^2-\nu^2)\}} + C = 0, \\ A'\mu\nu + B'\sqrt{\{(\mu^2-a'^2)(\nu^2-a'^2)\}} + C' = 0, \end{aligned}\right\} \quad (2)$$

where A, B, &c., are constants; and if we transform these expressions (2) to Cartesian co-ordinates, we have

$$\mu\nu = cx,$$

$$\sqrt{\{(a^2-\mu^2)(a^2-\nu^2)\}} = ab\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)},$$

$$\sqrt{\{(\mu^2-a'^2)(\nu^2-a'^2)\}} = a'b'\sqrt{\left(1-\frac{x^2}{a'^2}-\frac{y^2}{b'^2}\right)};$$

hence, from (2), we see at once that (1) represents a conic having double contact with the fixed conics. This equation (1) belongs to the system of conics whose equation is given at (1), Ex. 210. In a similar manner we can show that the differential equation

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-c^2)(a^2-\mu^2)(\mu^2-a'^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(c^2-\nu^2)(a^2-\nu^2)(a'^2-\nu^2)\}}} = 0, \quad (3)$$

belongs to the second system of conics considered in Ex. 210.

Again, we can show that

$$\frac{d\mu}{\sqrt{\{(\mu^2-c^2)(\mu^2-a^2)(a'^2-\mu^2)\}}} \pm \frac{d\nu}{\sqrt{\{(c^2-\nu^2)(a^2-\nu^2)(a'^2-\nu^2)\}}} = 0 \quad (4)$$

is the differential equation of the system of conics at (4), Ex. 210.

From these differential equations we see that the two conics of each system, which can be drawn through a point, make equal angles with the conics confocal with the given ones that pass through the point.

Also, if ϕ is the angle between the two conics of the system (1), we have

$$\tan \tfrac{1}{2} \phi = \sqrt{\left\{\frac{(\mu^2 - c^2)(a^2 - \nu^2)(\nu^2 - a'^2)}{(c^2 - \nu^2)(a^2 - \mu^2)(\mu^2 - a'^2)}\right\}},$$

and similar values for the other two systems.

213. If the normal at a point P of a conic of the system (4) in the preceding example touches the conic $\mu = a$, show the locus of P is the bicircular quartic whose equation to rectangular axes is

$$(x^2 + y^2)^2 - (b'^2 + 2a^2)\, x^2 - (a'^2 + 2b^2)\, y^2 + a^4 + 2a^2 a'^2 + c^4 = 0.$$

214. If a conic of the system (3), Ex. 212, cut orthogonally a conic of the system (4) in the same example, show that the locus of their intersection consists of the two circular cubics

$$cx\,(x^2 + y^2 \pm cx - a^2 - b'^2) \pm a^2 a'^2 = 0.$$

215. If two conics of the system (4), Ex. 212, cut each other at right angles, show that the locus of their intersection is the bicircular quartic

$$(x^2 + y^2)^2 - (a^2 + a'^2)\, x^2 - (b^2 + b'^2)\, y^2 + a^2 a'^2 + b^2 b'^2 = 0.$$

216. From the foci of the fixed conics tangents are drawn to the systems of conics (1), (3), (4) Ex. 212; show that the

equations of the loci of their points of contact are, respectively,

$$x^2 + y^2\left(1 + \frac{c^2}{b^2} + \frac{c^2}{b'^2}\right) - c^2 = 0,$$

$$\frac{x^2}{a^2 a'^2} + \frac{y^2}{b^2 b'^2} = 0,$$

$$c^2 y^2 + b^2 b'^2 = 0.$$

217. Normals are drawn from the foci of the fixed conics to the systems of conics referred to in the preceding example; show that the loci of their feet are, respectively,

$$x^2 + y^2 - a^2 - b'^2 = 0,$$

$$x^2 (x^2 + y^2 - a^2 - a'^2 - c^2) + a^2 a'^2 = 0,$$

$$(x^2 + y^2)^2 - (a^2 + a'^2)\, x^2 - (a^2 + b'^2)\, y^2 + a^2 a'^2 = 0.$$

218. Normals are drawn from the centre of the fixed conics to the systems of conics (1), (3), Ex. 212; show that the loci of their feet are, respectively,

$$(x^2 + y^2)^2 - (a^2 + b'^2)\, x^2 - (b^2 + b'^2)\, y^2 + b^2 b'^2 = 0,$$

$$(x^2 + y^2)^2 - (a^2 + a'^2)\, x^2 - (a^2 + b'^2)\, y^2 + a^2 a'^2 = 0.$$

219. Show that the locus of the vertices of the system of conics (4), Ex. 212, is

$$(x^2 + y^2)^3 - (a^2 + a'^2)(x^2 + y^2)^2 + 2c^2 y^2 (x^2 + y^2) + a^2 a'^2 x^2 + b^2 b'^2 y^2 = 0.$$

220. Show that the loci of the vertices of the systems of conics (1), (2), Ex. 212, are, respectively,

$$c^2 y^4 - c^2 y^2 (b^2 + b'^2) - b^2 b'^2 (x^2 - c^2) = 0,$$

$$c^2 x^4 - c^2 x^2 (a^2 + a'^2) + a^2 a'^2 (y^2 + c^2) = 0.$$

221. Tangents are drawn from the centre of the fixed conics to the systems of conics (1), (3), Ex. 212; show that the loci of the points of contact are, respectively,

$$(a^2a'^2y^2 + b^2b'^2x^2)(x^2 + y^2) - c^4x^2y^2 - c^2b^2b'^2x^2 = 0,$$

$$(a^2a'^2y^2 + b^2b'^2x^2)(x^2 + y^2) - c^4x^2y^2 + c^2a^2a'^2y^2 = 0.$$

222. If λ_1, λ_2 are the roots of the equation

$$\frac{x^2}{\lambda - a} + \frac{y^2}{\lambda - b} + \frac{z^2}{\lambda - c} = 0, \qquad (1)$$

where x, y, z are trilinear co-ordinates, show that the differential equations

$$\frac{d\lambda_1}{\sqrt{\{(\lambda_1-b)(\lambda_1-c)(\lambda_1-\alpha)(\lambda_1-\beta)\}}} \pm \frac{d\lambda_2}{\sqrt{\{(\lambda_2-b)(\lambda_2-c)(\lambda_2-\alpha)(\lambda_2-\beta)\}}} = 0,$$

$$\frac{d\lambda_1}{\sqrt{\{(\lambda_1-c)(\lambda_1-a)(\lambda_1-\alpha)(\lambda_1-\beta)\}}} \pm \frac{d\lambda_2}{\sqrt{\{(\lambda_2-c)(\lambda_2-a)(\lambda_2-\alpha)(\lambda_2-\beta)\}}} = 0,$$

$$\frac{d\lambda_1}{\sqrt{\{(\lambda_1-a)(\lambda_1-b)(\lambda_1-\alpha)(\lambda_1-\beta)\}}} \pm \frac{d\lambda_2}{\sqrt{\{(\lambda_2-a)(\lambda_2-b)(\lambda_2-\alpha)(\lambda_2-\beta)\}}} = 0,$$

represent the three systems of conics having double contact with the two fixed conics of the system (1), corresponding to the values

$$\lambda = \alpha, \quad \lambda = \beta.$$

223. A variable conic has double contact with two fixed conics; show that its director circle cuts orthogonally one or other of three fixed circles.

Also show that each of these fixed circles passes through the extremities of a diagonal of the quadrilateral formed by the common tangents of the two fixed conics.

224. In the preceding example, show that the variable director circle has double contact with one or other of three bicircular quartics, which reduce to cubics if one of the fixed conics becomes a parabola.

225. A parabola has double contact with a fixed conic, and touches a fixed line; to find the envelope of its directrix.

Let the tangential equation of the fixed conic referred to its axes be

$$a^2\lambda^2 + b^2\mu^2 - \nu^2 = 0, \qquad (1)$$

then

$$a^2\lambda^2 + b^2\mu^2 - \nu^2 + (\alpha\lambda + \beta\mu + \nu)^2 = 0 \qquad (2)$$

represents a parabola having double contact with (1).

Now, the directrix of (2) is (*Conics*, Art. 294)

$$2\alpha x + 2\beta y - (a^2 + b^2 + \alpha^2 + \beta^2) = 0, \qquad (3)$$

and if (2) touch the fixed line $lx + my + n = 0$, we have

$$l\alpha + m\beta + n \pm\sqrt{(n^2 - a^2l^2 - b^2m^2)} = 0. \qquad (4)$$

But the envelope of (3), subject to the condition (4), is found to be

$$(l^2 + m^2)(x^2 + y^2 - a^2 - b^2) - \{lx + my + n \pm\sqrt{(n^2 - a^2l^2 - b^2m^2)}\}^2 = 0,$$

which represents two parabolæ having double contact with the director circle of the fixed conic (1).

226. In the preceding example, show that the locus of the foci of the parabola consists of two circular cubics with nodes (see Ex. 87).

227. An equilateral hyperbola has double contact with the fixed conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and passes through the fixed point x', y'; show that the locus of its centre is the bicircular quartic

$$\{a^2b^2(x^2+y^2)-(a^2+b^2)(b^2xx'+a^2yy')\}^2$$
$$-(a^2+b^2)(b^2x'^2+a^2y'^2-a^2b^2)(b^4x^2+a^4y^2)=0.$$

228. Show that an infinite number of equilateral hyperbolæ can be described to have double contact with the fixed conics whose equations to rectangular axes are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

$$a^2x^2+b^2y^2-2(a^2+b^2)(\alpha x+\beta y)+(a^2+b^2)(\alpha^2+\beta^2)+a^2b^2=0.$$

229. An equilateral hyperbola has double contact with the parabola $y^2-4mx=0$, and passes through the fixed point x', y'; show that the locus of its centre is the conic

$$(y'^2-4mx')(y^2+4m^2)-(yy'+2mx-2mx'+4m^2)^2=0.$$

XI.—RELATIONS OF A CIRCLE AND A CONIC.

230. A circle S intersects a conic in four points; to show that perpendiculars at these points to the radii vectores drawn from a focus are all touched by a circle S'.

Referring the conic to one of its foci, its polar equation is

$$r(1 + e\cos\theta) = l; \qquad (1)$$

and if the equation of S is

$$r^2 - 2r(\alpha\cos\theta + \beta\sin\theta) + k^2 = 0, \qquad (2)$$

we have for their intersection, eliminating r between (1) and (2),

$$l^2 - 2l(1 + e\cos\theta)(\alpha\cos\theta + \beta\sin\theta) + k^2(1 + e\cos\theta)^2 = 0,$$

or
$$l^2 + k^2 + (k^2e^2 - 2le\alpha)\cos^2\theta + 2(k^2e - l\alpha)\cos\theta$$
$$- 2l\beta\sin\theta - 2le\beta\sin\theta\cos\theta = 0. \qquad (3)$$

Now, the equation of the perpendicular to a focal radius vector at its extremity is

$$x\cos\theta + y\sin\theta = r = \frac{l}{1 + e\cos\theta};$$

and if this line touch the circle

$$(x - \alpha')^2 + (y - \beta')^2 - r'^2 = 0,$$

we must have

$$(\alpha' \cos\theta + \beta' \sin\theta - r')(1 + e\cos\theta) - l = 0,$$

or
$$l + r' - e\alpha' \cos^2\theta + (er' - \alpha')\cos\theta$$
$$- \beta' \sin\theta - e\beta' \sin\theta\cos\theta = 0. \quad (4)$$

But it is easily seen that the relations (3) and (4) will coincide if

$$l\alpha' = 2l\alpha - ek^2, \quad \beta' = 2\beta, \quad lr' = k^2;$$

hence we see that, being given S in the form

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0,$$

the equation of S' is

$$\left(x - 2\alpha + \frac{ek^2}{l}\right)^2 + (y - 2\beta)^2 = \frac{k^4}{l^2}.$$

We can also show in the same way that the lines bisecting the focal radii vectores at right angles are all touched by the circle

$$\left(x - \alpha + \frac{ek^2}{2l}\right)^2 + (y - \beta)^2 = \frac{k^4}{4l^2}. \quad (5)$$

From this we can readily obtain a proof of Ex. 19; for when the curve is a parabola, $e = 1$, and the circle (5) then is satisfied by the co-ordinates of the centre of S.

231. If the circle S in the preceding example cuts a fixed circle orthogonally, show that the circle S' cuts a fixed line under a constant angle.

232. To show that the algebraic sum of the reciprocals of the common tangents of a circle and a conic is equal to zero.

The equation of the conic being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

the equation of a tangent is

$$x \cos\omega + y \sin\omega - \sqrt{(a^2 \cos^2\omega + b^2 \sin^2\omega)} = 0 \; ; \qquad (1)$$

then if p is the perpendicular on this tangent from the fixed point x', y', we know that $\frac{d\omega}{dp}$ is the reciprocal of the common tangent of the conic and a circle whose centre is x', y', and radius p. Hence, considering the four common tangents of the conic and circle, p is the same for all, and we have

$$\Sigma \frac{1}{t} = \frac{d}{dp} \Sigma\omega. \qquad (2)$$

But if we express that the line (1) touches a circle whose centre is x', y', and radius p, we get

$$x' \cos\omega + y' \sin\omega - p - \sqrt{(a^2 \cos^2\omega + b^2 \sin^2\omega)} = 0; \quad (3)$$

and putting, then, $e^{\omega\sqrt{-1}} = z$ in this equation (3), it becomes

$$\{(x' - y'\sqrt{-1})z^2 - 2pz + x' + y'\sqrt{-1}\}^2 - \{c^2 z^4 + 2(a^2 + b^2)z^2 + c^2\} = 0,$$

$$\text{or } \{(x' - y'\sqrt{-1})^2 - c^2\}z^4 + \ \ldots \ + (x' + y'\sqrt{-1})^2 - c^2 = 0, \ (4)$$

from which we readily deduce that $\Sigma\omega = 4\phi$, where

$$\tan 2\phi = \frac{2x'y'}{x'^2 - y'^2 - c^2} \, ;$$

hence, since $\Sigma\omega$ is independent of p, we see from (2) that

$$\Sigma \frac{1}{t} = 0.$$

If the circle touch the conic, by drawing two tangents to the curve indefinitely near one another, we obtain the following values for the lengths of the common tangents t_1, t_2,

$$t_1 = \tfrac{1}{2}(\rho - r)\, d\theta + \tfrac{1}{4}\rho \cot\phi d\theta^2,$$

$$t_2 = \tfrac{1}{2}(\rho - r)\, d\theta - \tfrac{1}{4}\rho \cot\phi d\theta^2,$$

where r is the radius of the circle, ρ the radius of curvature of the conic at the point of contact, and ϕ the angle which the diameter of the conic at the same point makes with the curve; hence, by proceeding to the limit, in this case $\frac{1}{t_1} \pm \frac{1}{t_2}$ is replaced by $$\frac{2\rho \cot\phi}{(\rho - r)^2}.$$

233. To show that a circle meets a conic at angles the sum of whose co-tangents is equal to zero.

If a circle of fixed centre and variable radius r meet the curve at an angle ψ, we have

$$\cot\psi = \frac{r d\omega}{dr},$$

where ω is the angle which the radius makes with a fixed line; hence, for the four points of intersection, since r is the same for all,

$$\Sigma \cot\psi = \frac{rd}{dr}\Sigma\omega. \qquad (1)$$

But since a pair of chords of intersection of the conic and circle are equally inclined to an axis of the conic, it easily follows that $\Sigma\omega$ is constant; hence, from (1),

$$\Sigma \cot\psi = 0.$$

If the circle touch the conic, the sum of two co-tangents must be replaced by $\frac{2r^2 \cot\phi}{(\rho - r)^2}$, where r, ρ, ϕ have the same meaning as in the preceding example.

234. If Σ is the sum of the squares of the lengths of the six chords of intersection of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and the circle

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0,$$

to show that

$$\Sigma = 16 \left\{ r^2 - \frac{(b^4\alpha^2 + a^4\beta^2)}{(a^2 - b^2)^2} \right\}.$$

Eliminating y between the equations of the circle and conic, we obtain

$$c^4x^4 - 4a^2c^2\alpha x^3 + 2a^2\{(3a^2 - b^2)\,\alpha^2 + (a^2 + b^2)\,\beta^2 - c^2r^2 + b^2c^2\}\,x^2 + \ldots = 0,$$

where $c^2 = a^2 - b^2$; hence from this equation we have

$$c^4\Sigma\,(x_1 - x_2)^2 = 16a^2\{b^2\alpha^2 - (a^2 + b^2)\,\beta^2 + c^2r^2 - b^2c^2\};$$

and similarly, by eliminating x, we obtain from the equation in y

$$c^4\Sigma\,(y_1 - y_2)^2 = 16\,b^2\{a^2\beta^2 - (a^2 + b^2)\,\alpha^2 - c^2r^2 + a^2c^2\};$$

hence

$$\Sigma = \Sigma\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$$

$$= 16 \left\{ r^2 - \frac{(b^4\alpha^2 + a^4\beta^2)}{c^4} \right\}.$$

235. If Σ is the sum of the squares of the lengths of the six chords of intersection of a circle and the parabola $y^2 - px = 0$, show that $\Sigma = 4\,(\delta^2 - p^2)$, where δ is the intercept which the circle makes on the axis of the parabola.

236. To find the sum of the angles (s) at which the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$$

cuts the conic $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

If $\Sigma\omega$ and $\Sigma\theta$ are the sums of the angles which the tangents to the conic and circle at their intersection make, respectively, with the axis of x, we have

$$s = \Sigma\omega - \Sigma\theta.$$

But since a pair of chords of intersection of the conic and circle are equally inclined to the axis of x, $\Sigma\theta = 0$; therefore

$$s = \Sigma\omega. \qquad (1)$$

But substituting the co-ordinates of a point on the conic expressed in the form

$$x = \frac{a^2 \cos\omega}{\sqrt{(a^2\cos^2\omega + b^2\sin^2\omega)}}, \quad y = \frac{b^2 \sin\omega}{\sqrt{a^2\cos^2\omega + b^2\sin^2\omega)}}$$

in the equation of the circle, we get

$$\{a^2(a^2+k^2)\cos^2\omega + b^2(b^2+k^2)\sin^2\omega\}^2$$
$$- 4(a^2\alpha\cos\omega + b^2\beta\sin\omega)^2(a^2\cos^2\omega + b^2\sin^2\omega) = 0. \quad (2)$$

Putting, then, $e^{2\omega\sqrt{-1}} = z$ in this equation (2), it becomes

$$\{c^2(a^2+b^2+k^2)^2 - 4(a^2\alpha + b^2\beta\sqrt{-1})^2\}z^4 + \ldots$$
$$+ c^2(a^2+b^2+k^2)^2 - 4(a^2\alpha - b^2\beta\sqrt{-1})^2 = 0,$$

from which we obtain

$$e^{2\sqrt{-1}\Sigma\omega} = \frac{c^2(a^2+b^2+k^2)^2 - 4(a^4\alpha^2 - b^4\beta^2) + 8a^2b^2\alpha\beta\sqrt{-1}}{c^2(a^2+b^2+k^2)^2 - 4(a^4\alpha^2 - b^4\beta^2) - 8a^2b^2\alpha\beta\sqrt{-1}}; \quad (3)$$

hence from (1) and (3) we have at once

$$\tan s = \frac{8a^2b^2\alpha\beta}{c^2(a^2+b^2+k^2)^2 - 4(a^4\alpha^2 - b^4\beta^2)}.$$

237. If s is the sum of the angles at which the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$$

cuts the parabola $\qquad y^2 - 4mx = 0,$

show that $\qquad \tan s = \dfrac{4m\beta}{k^2 + 2m\alpha - 3m^2};$

hence also show that when s is given, the circle cuts orthogonally a fixed circle having its centre on the directrix and passing through the focus.

238. To find the equation which determines the lengths of the perpendiculars from the origin on the common tangents of the circle

$$(x-\alpha)^2 + (y-\beta)^2 - r^2 = 0,$$

and the conic $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1 = 0.$

Expressing that the tangent

$$x\cos\omega + y\sin\omega - \sqrt{(a^2\cos^2\omega + b^2\sin^2\omega)} = 0$$

of the conic touches the circle, we obtain

$$\alpha\cos\omega + \beta\sin\omega - (p + r) = 0, \qquad (1)$$

where $\qquad p^2 = a^2\cos^2\omega + b^2\sin^2\omega. \qquad (2)$

But from (2) we have

$$c\cos\omega = \sqrt{(p^2 - b^2)}, \quad c\sin\omega = \sqrt{(a^2 - p^2)};$$

hence (1) becomes

$$\alpha \sqrt{(p^2 - b^2)} + \beta \sqrt{(a^2 - p^2)} - c(p + r) = 0. \quad (3)$$

Clearing, then, (3) of radicals, we have the required equation. From this result we find that the product of the four perpendiculars is equal to

$$\frac{1}{\rho^2 \rho'^2}\{c^4 r^4 - 2c^2 r^2 (a^2\beta^2 - b^2\alpha^2) + (b^2\alpha^2 + a^2\beta^2)^2\},$$

where ρ, ρ' are the distances of the centre of the circle from the foci. We find also that the algebraic sum of the perpendiculars is

$$\frac{4rc^2}{\rho^2\rho'^2}(\alpha^2 - \beta^2 - c^2).$$

239. If Δ is the area of the triangle formed by the centre of a conic and the points of contact of a common tangent of the conic and a circle, show that

$$\Delta = \tfrac{1}{2}\rho\rho' \sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta \sin \tfrac{1}{2}\gamma,$$

where ρ, ρ' are the distances of the centre of the circle from the foci of the conic, and α, β, γ are the angles which the common tangent makes with the three other common tangents.

240. Show that the product of the lengths of the common tangents of the circle

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0$$

and the conic $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is equal to $$\frac{(h_1^2 - h_2^2)^2 (h_2^2 - h_3^2)^2 (h_3^2 - h_1^2)^2}{c^4 r^4 - 2c^2 r^2 (a^2\beta^2 - b^2\alpha^2) + (a^2\beta^2 + b^2\alpha^2)^2},$$

where h_1^2, h_2^2, h_3^2 are the roots of the equation

$$\frac{\alpha^2}{a^2 - h^2} + \frac{\beta^2}{b^2 - h^2} + \frac{r^2}{h^2} - 1 = 0.$$

241. Show that the product of the lengths of the common tangents of the circle

$$(x - \alpha)^2 + (y - \beta)^2 - r^2 = 0$$

and the parabola $y^2 - px = 0$

is equal to $\frac{1}{4}p^4 (k_1 - k_2)^2 (k_2 - k_3)^2 (k_3 - k_1)^2$,

where k_1, k_2, k_3 are the roots of the equation

$$4m^2k^3 + 4m(\alpha + m)k^2 + (r^2 + 4m\alpha - \beta^2)k + r^2 = 0.$$

242. To find the angles at which a circle cuts a conic in terms of the angles which their chords of intersection make with an axis of the conic.

The equations of the conic and circle referred to trilinear co-ordinates being written

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0, \quad \beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 0, \quad (1)$$

where α, β, γ are the sides of a triangle formed by their intersection, the tangents to these curves at α, β are

$$l\beta + m\alpha = 0, \quad \beta \sin A + \alpha \sin B = 0; \quad (2)$$

and if ϕ is the angle between the lines (2), we easily find

$$\cot\phi = \sin C \frac{(l \cos B - m \cos A)}{l \sin B - m \sin A}. \quad (3)$$

But substituting

$$x \cos\alpha + y \sin\alpha - p_1, \quad x \cos\beta + y \sin\beta - p_2, \quad x \cos\gamma + y \sin\gamma - p_3$$

for a, β, γ, respectively, in $l\beta\gamma + m\gamma a + na\beta$, and equating the result to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1,$$

we obtain, by comparison of the coefficients of x^2, y^2 and xy,

$$l(a^2 \cos\beta \cos\gamma - b^2 \sin\beta \sin\gamma) + m(a^2 \cos\gamma \cos a - b^2 \sin a \sin\gamma)$$
$$+ n(a^2 \cos a \cos\beta - b^2 \sin a \sin\beta) = 0,$$

$$l \sin(\beta + \gamma) + m \sin(\gamma + a) + n \sin(a + \beta) = 0,$$

from which we may assume l, m, n, respectively, proportional to

$$(a^2 \cos^2 a + b^2 \sin^2 a) \sin(\beta - \gamma), \quad (a^2 \cos^2\beta + b^2 \sin^2\beta) \sin(\gamma - a),$$
$$(a^2 \cos^2\gamma + b^2 \sin^2\gamma) \sin(a - \beta); \qquad (4)$$

hence, substituting these values (4) in (3), and putting for A, B, C their values in terms of a, β, γ, we obtain, after some reductions,

$$\cot\phi = \frac{2b^2 + c^2\{\sin^2 a + \sin^2\beta + \sin^2\gamma - \sin^2(a + \beta - \gamma)\}}{2c^2 \sin(\gamma - a) \sin(\beta - \gamma) \sin(a + \beta)}. \quad (5)$$

Since the pairs of chords of intersection of a conic and a circle are equally inclined to an axis of the conic, we obtain the expressions for the other angles of intersection by changing the signs of a, β, γ in (5). The most symmetrical expression is obtained by altering the sign of γ in (5), when we have

$$\cot\phi = \frac{2b^2 + c^2\{\sin^2 a + \sin^2\beta + \sin^2\gamma - \sin^2(a + \beta + \gamma)\}}{2c^2 \sin(a + \beta) \sin(\beta + \gamma) \sin(\gamma + a)}. \quad (6)$$

If we put $a = \beta = \gamma$ in (6), we obtain an expression for the angle at which the osculating circle at any point of a conic cuts the curve again.

For the parabola we have

$$\cot\phi = \frac{\sin^2\alpha + \sin^2\beta + \sin^2\gamma - \sin^2(\alpha+\beta+\gamma)}{2\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha)}.$$

From the expressions obtained above for the angles of intersection of a conic and a circle we can easily verify the theorem of Ex. 233.

243. Let

$$S = (x-\alpha)^2 + (y-\beta)^2 - r^2,$$

$$S' = (x-\alpha)^2 + (y+\beta)^2 - r^2;$$

then, if $SS' + 4\lambda(b^2x^2 + a^2y^2 - a^2b^2)$

breaks up into two circles, show that λ is determined by the equation

$$\frac{a^2b^2}{\beta^2 - \lambda a^2} - \frac{b^2\alpha^2}{\beta^2 - \lambda(a^2-b^2)} + \frac{r^2}{\lambda} - a^2 = 0.$$

244. Show that the product of the perpendiculars from the origin on the tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

at the points of intersection with the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$$

is equal to $$\frac{a^4b^4c^2}{\sqrt{(P^2+Q^2)}},$$

where $$P = c^2(a^2+b^2+k^2)^2 - 4(a^4\alpha^2 - b^4\beta^2),$$

$$Q = 8a^2b^2\alpha\beta.$$

245. If p is the perpendicular from the centre of a conic at a point of intersection with a circle, and ϕ the angle under which the conic and circle cut at the same point, show that, taking the four points of intersection,

$$\Sigma \frac{p}{\sin\phi} = 0.$$

246. If four tangents to a conic are all touched by a circle, show that their points of contact are situated on a conic having double contact with the director circle at the points where it is intersected by the polar of the centre of the circle with regard to the given curve.

XII.—CIRCLES RELATED TO A CONIC.

247. To find the equation of the circle described on a chord of a conic as diameter.

Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \qquad (1)$$

be the equation of the conic, and

$$lx + my - 1 = 0 \qquad (2)$$

that of the chord.

Now, the equation of the circle described on the line joining the points $x_1 y_1$, $x_2 y_2$, as diameter, is

$$x^2 + y^2 - (x_1 + x_2)\,x - (y_1 + y_2)\,y + x_1x_2 + y_1y_2 = 0,$$

or, in terms of the eccentric angles α, β of two points on the conic (1),

$$x^2 + y^2 - a\,(\cos\alpha + \cos\beta)\,x - b\,(\sin\alpha + \sin\beta)\,y$$
$$+ a^2\cos\alpha\cos\beta + b^2\sin\alpha\sin\beta = 0. \quad (3)$$

But, comparing

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta) = 0$$

with (2), we obtain

$$\cos\tfrac{1}{2}(\alpha + \beta) = al\cos\tfrac{1}{2}(\alpha - \beta), \quad \sin\tfrac{1}{2}(\alpha + \beta) = bm\cos\tfrac{1}{2}(\alpha - \beta);$$

hence (3) becomes

$$(a^2l^2 + b^2m^2)(x^2 + y^2) - 2a^2lx - 2b^2my + a^2 + b^2$$
$$- a^2b^2(l^2 + m^2) = 0. \quad (4)$$

By writing (4) in the form

$$(l^2+m^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\frac{c^2}{a^2b^2}(lx+my-1)\left\{lx-my-\frac{(a^2+b^2)}{c^2}\right\}=0,$$

we see that the circle meets the conic again on the line

$$lx-my-\frac{(a^2+b^2)}{c^2}=0. \qquad (5)$$

248. Writing the equation (4), in the preceding example, in the form

$$(x-la^2)^2+(y-mb^2)^2+(a^2l^2+b^2m^2-1)(x^2+y^2-a^2-b^2)=0, \ (1)$$

we see that this circle never meets the director circle in real points, except in the case when it touches it. It also follows from this equation (1) that the pole of a chord is a limiting point of the director circle and the circle described on the chord as diameter.

If we seek the length t of a common tangent of the director circle and the circle (4) in the preceding example, we easily find

$$t=\frac{\sqrt{\{(a^2+b^2)(1-a^2l^2-b^2m^2)\}}\pm ab\sqrt{-1}\sqrt{(l^2+m^2)}}{\sqrt{(a^2l^2+b^2m^2)}};$$

hence, being given the sum or difference of the external and internal common tangents, the chord is either parallel to a fixed line, or touches a concentric, similar, and similarly situated conic. Also given the product of these common tangents, the envelope of the chord is a concentric conic; and given their ratio, the envelope is a confocal conic. In all these cases, of course, the given conic is a hyperbola.

249. The circles of the system (4), Ex. 247, touch the given conic; show that the envelope of the chords is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{c^4}{(a^2 + b^2)^2} = 0.$$

250. Show that the circles described on parallel chords of a conic as diameters have double contact with a fixed conic.

251. If we express that the circle (4), Ex. 247, cuts orthogonally the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0, \qquad (1)$$

we obtain

$$a^2(k^2 - b^2)l^2 + b^2(k^2 - a^2)m^2 - 2a^2\alpha l - 2b^2\beta m + a^2 + b^2 = 0, \quad (2)$$

showing that the chord in the same case touches a fixed conic, of which (2) is the tangential equation. If we have

$$\frac{a^2\alpha^2}{k^2 - b^2} + \frac{b^2\beta^2}{k^2 - a^2} = a^2 + b^2,$$

the conic (2) breaks up into two points.

Now this relation (3) is the condition that the circle (1) should be the polar circle of a triangle formed by two tangents and their chord of contact (see Ex. 79, (2)).

If l, m are the co-ordinates of the line joining the two points into which (2) breaks up, we have, by differentiation,

$$(k^2 - b^2)\,l = \alpha, \quad (k^2 - a^2)\,m = \beta, \quad la^2\alpha + mb^2\beta = a^2 + b^2; \quad (4)$$

and if we put $$l = \frac{x'}{a^2}, \quad m = \frac{y'}{b^2}$$

in these equations (4), where x', y' are the co-ordinates of the pole of the line, we find

$$\alpha\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=\frac{x'}{a^2}\left(a^2+b^2+\frac{c^2y'^2}{b^2}\right),$$

$$\beta\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=\frac{y'}{b^2}\left(a^2+b^2-\frac{c^2x'^2}{a^2}\right),$$

$$k^2\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}\right)=a^2+b^2+\frac{b^2}{a^2}x'^2+\frac{a^2}{b^2}y'^2.$$

These values of α, β, k^2 (see Ex. 78, (1)), show that the circle (1) is the polar circle of the triangle formed by the tangents from x', y' and their chord of contact. Thus we see that if the circle described on the chord as diameter cuts orthogonally the polar circle of a triangle formed by the tangents from a point P and their chord of contact, then the chord will pass through one or other of two fixed points lying on the polar of P.

If x_1, y_1 are the co-ordinates of one of these points, the co-ordinates x_2, y_2 of the other are

$$x_2=\frac{c^2}{a^2+b^2}\frac{x_1}{\frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}},\quad y_2=-\frac{c^2}{a^2+b^2}\frac{y_1}{\frac{x_1{}^2}{a^2}+\frac{y_1{}^2}{b^2}},$$

and the two points are reciprocally related.

252. For the equilateral hyperbola $x^2-y^2-a^2=0$ the equation (4), Ex. 247, becomes

$$(l^2-m^2)(x^2+y^2)-2lx+2my+a^2(l^2+m^2)=0;$$

hence, if two circles of this system cut each other orthogonally, we get

$$l^2l'^2-m^2m'^2-a^2(ll'+mm')=0,$$

which breaks up into the factors

$$ll' - mm' = 0, \quad ll' - mm' - a^2 = 0.$$

In the first case, the chords are rectangular, and in the second case, conjugate with respect to the curve. Thus we see that circles described on conjugate or rectangular chords of an equilateral hyperbola, as diameters, cut each other orthogonally.

253. Show that the circle described on the chord $lx+my-1=0$ of the equilateral hyperbola $x^2 - y^2 - a^2 = 0$ is the polar circle of the triangle formed by the tangents from the point $\frac{2}{l}, \frac{2}{m}$ and their chord of contact.

254. Show that the circle described on a chord of an equilateral hyperbola as diameter meets the curve again at the extremities of a diameter of the curve.

255. If the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0$$

passes through the extremities of a diameter of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

show that $$\alpha^2\left(1 + \frac{k^2}{b^2}\right) + \beta^2\left(1 + \frac{k^2}{a^2}\right) = 0.$$

256. Through two points A, B on a conic U a circle S is described, whose radius is equal to the semidiameter of U parallel to AB; show that two common tangents of S and U are parallel to one another.

257. Through a pair of points A, B on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

two circles are described to touch the curve elsewhere; show that their points of contact are at the extremities of a diameter, and that, if r, r' are their radii, and δ the distance between their centres,

$$\delta = \left(\frac{a^2 - b^2}{ab}\right) b', \quad r + r' = \frac{(a^2 + b^2)}{ab} b',$$

where b' is the length of the semidiameter parallel to AB. Show also that the length of their common tangent is equal to

$$\left(\frac{a^2 - b^2}{2ab}\right) d,$$

where d is the length of AB.

258. Show that the angle between the two circles referred to in the preceding example remains the same if the chord AB always touches a fixed concentric, similar, and similarly situated conic.

259. Through a pair of points on an equilateral hyperbola two circles are described to touch the curve; show that their radii are both equal to the semidiameter to one of the points of contact.

260. Through two points on a hyperbola two circles are described to touch the curve elsewhere; show that the sum of their radii is equal to the length of the diameter conjugate to that passing through the points of contact.

261. A circle S is described through the points where the line

$$lx + my - 1 = 0 \qquad (1)$$

meets the conic $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

so that these points subtend an an angle ϕ at the circumference; show that S meets the curve again at points lying on the line

$$(a^2-b^2)(lx-my)-(a^2+b^2)-2ab\cot\phi\sqrt{(a^2l^2+b^2m^2-1)}=0. \quad (2)$$

262. If, in the preceding example, one of the points where the line (1) meets the curve is fixed, show that the line (2) passes through a fixed point on the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(a^2+b^2)^2+4a^2b^2\cot^2\phi}{(a^2-b^2)^2}.$$

If the line (1) touches a concentric, similar, and similarly situated conic, show that the line (2) will touch another one.

263. A circle S passes through the centre of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\equiv U=0;$$

show that the product of the perpendiculars from the centre of U on a pair of chords of intersection of S and U is equal to

$$\frac{a^2b^2}{a^2-b^2}.$$

264. A circle touches two fixed tangents to a conic; show that a pair of its chords of intersection with the conic are parallel to given lines.

265. A circle passes through two fixed points on a conic; show that the extremities of one of the diagonals of the quadrilateral formed by the common tangents of the circle and the conic lie on a given confocal conic.

266. Tangents are drawn from the point whose elliptic co-ordinates are μ, ν to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0;$$

show that the equation of one of the circles touching the curve and these tangents is

$$x^2 + y^2 - 2\frac{\nu}{c}Px - \frac{2}{bc}\sqrt{(c^2 - \nu^2)(aP - c^2)}\,y + \nu^2 - a^2 - c^2 + 2aP = 0,$$

where $$P = \frac{a\sqrt{(\mu^2 - c^2)} - b\mu}{\sqrt{(\mu^2 - c^2)} - b}, \quad a^2 - b^2 = c^2$$

(see Ex. 44, (6)).

Hence show that if the intersection of the tangents lies on a confocal ellipse, the locus of the centre of the circle is an ellipse.

267. A circle Σ cuts the circle $S = x^2 + y^2 - k^2 = 0$ orthogonally, and touches the conic

$$V \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0\,;$$

show that the equation of the reciprocal polar of the locus of the centre of Σ with respect to S is the trinodal quartic

$$(k^2V - cS)^2 + 4k^2(gx + fy + c)^2 S = 0.$$

268. Through the four points on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

whose eccentric angles are $\alpha, \beta, -\alpha, -\beta$, a circle is described; if θ is the angle subtended by α, β at the centre of the circle, show that

$$\tan\tfrac{1}{2}\theta = \frac{a}{b}\tan\tfrac{1}{2}(\alpha - \beta).$$

269. A circle S described on a chord of the equilateral hyperbola $x^2 - y^2 - a^2 = 0$ as diameter cuts orthogonally the circle

$$x^2 + y^2 - 2\alpha x - 2\beta y + k^2 = 0\,;$$

show that the locus of the centre of S is the nodal cubic

$$2(\alpha x + \beta y)(x^2 - y^2) - (a^2 + k^2)x^2 - (a^2 - k^2)y^2 = 0.$$

270. A circle S described on a chord of the parabola $y^2 - px = 0$ as diameter cuts orthogonally the circle

$$x^2 + y^2 - k^2 = 0\,;$$

show that the locus of the centre of S breaks up into two parabolæ.

271. PP', QQ' are fixed chords of a conic parallel to an axis of the curve; if a circle through P, P' intersect a circle through Q, Q' on the curve, show that the distance between their centres is constant.

272. A conic passes through two fixed points and touches two fixed lines; show that the envelope of the director circle consists of two nodal bicircular quartics, the node common to each being the intersection of the fixed lines.

XIII.—RECIPROCAL TRIANGLES.

273. To find the condition that a conic S_1 should circumscribe a triangle Δ, whose reciprocal triangle with regard to S is self-conjugate with regard to S_2.

If we take the reciprocal conic of S_2 with regard to S, this conic must have the triangle Δ for a self-conjugate triangle, and, therefore, satisfy an invariant relation with S_1. Referring the conics to the common self-conjugate triangle of S and S_1, we may write

$$S \equiv x^2 + y^2 + z^2,$$

$$S_1 \equiv ax^2 + by^2 + cz^2,$$

$$S_2 \equiv (a', b', c', f', g', h')(x, y, z)^2,$$

and the reciprocal of S_2 with regard to S is then

$$(A', B', C', F', G', H')(x, y, z)^2 = 0,$$

where A, B, &c., are the coefficients in the tangential equation of S_2. The required condition is then found to be (*Conics*, Art. 375),

$$aa' + bb' + cc' = 0. \qquad (1)$$

But if Θ_{133}, Θ_{233}, Θ_{123}, are the coefficients of ln^2, mn^2, lmn in the discriminant of

$$lS_1 + mS_2 + nS,$$

we have

$$\Theta_{133} = a + b + c, \quad \Theta_{233} = a' + b' + c',$$

$$\Theta_{123} = a'(b + c) + b'(c + a) + c'(a + b);$$

hence, from (1),

$$\Theta_{133}\,\Theta_{233} = \Delta\Theta_{123}, \qquad (2)$$

where Δ is the discriminant of S. This condition (2) is symmetrical between the coefficients of S_1 and S_2, and, therefore, when it is satisfied, it follows also that triangles can be inscribed in S_2, whose reciprocals with regard to S shall be self-conjugate with regard to S_1.

If S_1 and S_2 are the circles

$$S_1 \equiv x^2 + y^2 - 2\alpha_1 x - 2\beta_1 y + k_1^2,$$

$$S_2 \equiv x^2 + y^2 - 2\alpha_2 x - 2\beta_2 y + k_2^2,$$

and S is the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

the condition (2) gives

$$k_1^2 k_2^2 - 2(a^2\alpha_1\alpha_2 + b^2\beta_1\beta_2) + a^4 + b^4 = 0. \qquad (3)$$

Since a self-conjugate triangle is its own reciprocal with regard to a conic, it follows that the polar and circumscribing circles of such a triangle are connected by the relation (3).

The same relation will also connect the polar and circumscribing circles of a circumscribed triangle Δ with the circumscribing and polar circles, respectively, of the triangle formed by the points of contact of the sides of Δ; for these triangles are evidently reciprocal with regard to the curve.

274. If in the preceding example the conic S and the circle S_1 are fixed, show that the circle S_2 cuts orthogonally the fixed circle

$$k_1^2(x^2 + y^2) - 2a^2\alpha_1 x - 2b^2\beta_1 y + a^4 + b^4 = 0.$$

275. Show that the intersection of the perpendiculars of

a triangle formed by three tangents to an equilateral hyperbola and the centre of the circle passing through the points of contact of the tangents are conjugate with respect to the curve. Show also that the centre of the circle circumscribing the first triangle and the intersection of the perpendiculars of the second are conjugate with respect to the curve.

276. If we suppose the conics S_1 and S_2 in Ex. 273 to coincide, we see that if a conic U be such that an inscribed and a self-conjugate triangle are reciprocal with regard to a conic V, we must have

$$\Theta^2 = 2\Delta\Theta', \qquad (1)$$

where

$$\Delta k^3 + \Theta k^2 + \Theta' k + \Delta' \qquad (2)$$

is the discriminant of $V + kU$. The relation (1) expresses that the sum of the squares of the roots of (2) is equal to nothing. In this case, therefore, the conics U and V cannot intersect in more than two real points.

277. In the same way as at Ex. 273 we can find the relation which must exist, if a conic S_1 touch the sides of a triangle whose reciprocal with regard to S is self-conjugate with regard to S_2. This relation is $\Theta_{311}\,\Theta_{322} = \Phi$, where Φ is the invariant which corresponds in tangential co-ordinates to Θ_{123}.

278. If the conic S_1 circumscribe one triangle and S_2 another, and if these triangles are reciprocal with regard to a conic S_3, we find, with the notation of Ex. 273,

$$(\Theta_{133}\,\Theta_{233} - \Delta\Theta_{123})^2 = 4\Delta^2(\Theta_{311}\,\Theta_{322} - \Phi).$$

279. If we suppose the conics S_1 and S_2 in the preceding example to coincide, we obtain the condition that a conic U should circumscribe two triangles which are reciprocal with

regard to a conic V. This condition breaks up into the factors

$$\Theta = 0, \qquad (1)$$

$$\Theta^3 - 4\Delta\Theta\Theta' + 8\Delta'\Delta^2 = 0. \qquad (2)$$

The condition (1) only relates to the case when the triangles coincide and become self-conjugate with regard to V.

It may be observed that when (2) is satisfied it is possible to circumscribe about V an infinite number of quadrilaterals which have the extremities of two of their diagonals on U, and we can see how this is the case; for, if we suppose two sides of the quadrilateral to coincide, it will become a triangle formed by two tangents of V and their chord of contact, and such a triangle is its own reciprocal with regard to V.

280. If t is the length of the tangent drawn from the centre of an equilateral hyperbola to the circumscribing circle of a triangle whose area is Δ, and t', Δ' are the corresponding values for the reciprocal triangle, to show that

$$\frac{t^2}{\Delta} = \frac{t'^2}{\Delta'}.$$

Since the polar of the point x', y' with regard to the equilateral hyperbola $x^2 - y^2 - a^2 = 0$ is $xx' - yy' - a^2 = 0$, and with regard to the circle $x^2 + y^2 - a^2 = 0$ is $xx' + yy' - a^2 = 0$, it follows that the reciprocals of a given triangle with regard to the hyperbola and circle are their mutual reflections with regard to the axis of x. It follows then that if the relation stated above is true for a circle, it will also be true for an equilateral hyperbola.

Let x_1y_1, x_2y_2, x_3y_3 be the co-ordinates of the vertices of one of the triangles, then we have (*Conics*, Art. 94),

$$2\Delta t^2 = \Sigma(x_1y_2 - y_1x_2)(x_3^2 + y_3^2) = \rho_1\rho_2\rho_2\,\Sigma\rho_1 \sin(\theta_2 - \theta_3), \quad (1)$$

transforming to polar co-ordinates; but if the circle is

$$x^2 + y^2 - a^2 = 0, \quad \rho_1 = \frac{a^2}{p_1}, \text{ \&c.}, \quad \theta_1 = \omega_1, \text{ \&c.},$$

for the reciprocal triangle; hence (1) gives

$$2\Delta t^2 = \frac{a^8}{(p_1 p_2 p_3)^2} \Sigma p_1 p_2 \sin(\omega_1 - \omega_2). \tag{2}$$

Now $\Sigma p_1 p_2 \sin(\omega_1 - \omega_2)$ is equal to $\frac{\Delta'}{2R'^2} t'^2$; thus (2) becomes

$$2\Delta t^2 = \frac{a^8}{(p_1 p_2 p_3)^2} \frac{\Delta' t'^2}{R'^2}. \tag{3}$$

Again, we have

$$2\Delta = \Sigma \rho_1 \rho_2 \sin(\theta_1 - \theta_2) = \frac{a^4}{(p_1 p_2 p_3)} \Sigma p_1 \sin(\omega_2 - \omega_3) = \frac{a^4}{(p_1 p_2 p_3)} \frac{\Delta'}{R'}. \tag{4}$$

Hence, eliminating $(p_1 p_2 p_3)$ between (3) and (4), we obtain the relation given above.

281. Let p_1 be the lengths of the tangents drawn from the centre of a circle or equilateral hyperbola to the polar and nine-point circles, respectively, of a triangle whose area is Δ, and let p', n', Δ' be the corresponding values for the reciprocal triangle, then show that

$$n^2 = \frac{\Delta}{2\Delta'} p'^2, \quad n'^2 = \frac{\Delta'}{2\Delta} p^2.$$

282. If two triangles are inscribed in a circle so as to be polar reciprocals with regard to an equilateral hyperbola, show that their areas are equal.

283. Let lines drawn from the centre of a conic to the vertices of a triangle whose area is Δ meet the sides of the triangle in L, M, N; if the area of the triangle LMN is

equal to A, and A', Δ' are the corresponding values for the reciprocal triangle, to show that $\frac{A}{\Delta} = \frac{A'}{\Delta'}$.

Let x_1y_1, x_2y_2, x_3y_3 be the vertices of one of the triangles; then if α is the angle between the polars of x_1y_1, x_2y_2 with regard to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

we have

$$\sin \alpha = \frac{p_1 p_2}{a^2 b^2} (x_1 y_2 - y_1 x_2) ; \tag{1}$$

but

$$\Sigma p_1 \sin \alpha = \frac{\Delta'}{R'} =, \text{ from (1)}, \frac{2\Delta\, p_1 p_2 p_3}{a^2 b^2} ; \tag{2}$$

therefore, from (1) and (2),

$$\frac{2R' \sin \alpha p_3}{\Delta'} = \frac{x_1 y_2 - y_1 x_2}{\Delta}. \tag{3}$$

Hence, if xyz, $x'y'z'$ are the areal co-ordinates of the origin with regard to the two triangles, we have, from (3), $\frac{x}{\Delta} = \frac{x'}{\Delta'}$, and by symmetry for the other sides,

$$\frac{y}{\Delta} = \frac{y'}{\Delta'}, \quad \frac{z}{\Delta} = \frac{z'}{\Delta'}. \tag{4}$$

Now it can be shown that

$$A = \frac{2\Delta\, xyz}{(x+y)(y+z)(z+x)}, \quad A' = \frac{2\Delta' x'y'z'}{(x'+y')(y'+z')(z'+x')} ;$$

hence, from (4) we have $\frac{A}{\Delta} = \frac{A'}{\Delta'}$.

284. If two triangles, reciprocal with regard to a conic U, are such that their centroids are conjugate with respect to V, show that a conic circumscribing either triangle, so that the tangent at each vertex is parallel to the opposite side, will pass through the centre of U.

285. Two triangles, polar reciprocals with regard to an equilateral hyperbola, are such that their circumscribing circles pass through the centre of the hyperbola; show that they are similar to one another.

XIV.—MISCELLANEOUS EXAMPLES.

286. A series of conics are circumscribed about a fixed quadrilateral; to show that the envelope of their director circles is a bicircular quartic, of which the intersections of the diagonals and opposite sides are foci.

Since the locus of the centre of the variable director circle is a conic passing through the intersection of the diagonals and opposite side of the quadrilateral, and since it also cuts orthogonally the circle passing through the same points, its envelope is a bicircular quartic, of which the intersection of the fixed circle and conic are foci. The fourth focus of the quartic, it is easy to see, is the centre of the equilateral hyperbola which circumscribes the quadrilateral.

We may obtain this result analytically, as follows:—

Let the equation of the conic referred to the fixed self-conjugate triangle be

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0\,; \qquad (1)$$

then the equation of the director circle is (*Conics*, Art. 383)

$$mn\,(\beta^2 + \gamma^2 + 2\beta\gamma\cos A) + nl\,(\gamma^2 + \alpha^2 + 2\gamma\alpha\cos B) + lm\,(\alpha^2 + \beta^2 + 2\alpha\beta\cos C) = 0\,;$$

or, if ρ_1, ρ_2, ρ_3 denote the distances of a point from the vertices of the triangle of reference,

$$mn\sin^2 A\rho_1^2 + nl\sin^2 B\rho_2^2 + lm\sin^2 C\rho_3^2 = 0. \qquad (2)$$

But if α', β', γ' are the co-ordinates of one of the vertices of the quadrilateral, we have

$$l\alpha'^2 + m\beta'^2 + n\gamma'^2 = 0,$$

and the envelope of (2), subject to this condition, is

$$\alpha' \sin A\rho_1 + \beta' \sin B\rho_2 + \gamma' \sin C\rho_3 = 0. \qquad (3)$$

If we suppose one of the points to go off to infinity, we see that if a conic pass through three fixed points, and have an asymptote parallel to a given line, the envelope of its director circle will be a circular cubic.

287. A conic is described through the intersections of the diagonals and opposite sides of the quadrilateral referred to in the preceding example, so as to have a focus on the bicircular quartic; to show that its corresponding directrix will pass through a vertex of the quadrilateral.

From (3), in the preceding example, the focus of the conic satisfies the relation

$$\alpha' \sin A\rho_1 + \beta' \sin B\rho_2 + \gamma'\rho_3 \sin C = 0\,;$$

but ρ_1, ρ_2, ρ_3 are evidently proportional to the perpendiculars p_1, p_2, p_3 on the corresponding directrix; hence we have

$$\alpha' \sin Ap_1 + \beta' \sin Bp_2 + \gamma' \sin Cp_3 = 0\,;$$

but this is evidently the condition that a line should pass through the point α', β', γ'.

288. Ten pairs of circles are described through six points on a conic; to show that the middle points of the centres of these pairs lie on a line.

Let the equation of the conic be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

and let ϕ_1, ϕ_2, &c., be the eccentric angles of the six points; then, if x_1, y_1, x_2, y_2, are the co-ordinates of the centres of the circles passing through the points ϕ_1, ϕ_2, ϕ_3, and ϕ_4, ϕ_5, ϕ_6, respectively, we have, from (11), Ex. 1.,

$$x_1 = \frac{c^2}{4a}\left\{\cos\phi_1 + \cos\phi_2 + \cos\phi_3 + \cos(\phi_1 + \phi_2 + \phi_3)\right\},$$

$$y_1 = -\frac{c^2}{4b}\left\{\sin\phi_1 + \sin\phi_2 + \sin\phi_3 - \sin(\phi_1 + \phi_2 + \phi_3)\right\},$$

$$x_2 = \frac{c^2}{4a}\left\{\cos\phi_4 + \cos\phi_5 + \cos\phi_6 + \cos(\phi_4 + \phi_5 + \phi_6)\right\},$$

$$y_2 = -\frac{c^2}{4b}\left\{\sin\phi_4 + \sin\phi_5 + \sin\phi_6 - \sin(\phi_4 + \phi_5 + \phi_6)\right\};$$

hence, if x, y are the co-ordinates of the middle point of $x_1 y_1$, $x_2 y_2$, we have

$$x = \frac{c^2}{8a}(P + 2\cos s\cos\theta), \qquad (1)$$

$$y = -\frac{c^2}{8b}(Q - 2\sin s\cos\theta), \qquad (2)$$

where $P = \Sigma\cos\phi$, $Q = \Sigma\sin\phi$, $2s = \Sigma\phi$,

and $\phi_1 + \phi_2 + \phi_3 - \phi_4 - \phi_5 - \phi_6 = 2\theta$.

Eliminating, then, θ between (1) and (2), we obtain the equation

$$ax\sin s - by\cos s - \tfrac{1}{8}c^2(P\sin s + Q\cos s) = 0. \qquad (3)$$

But this evidently represents a line which is the same for every one of the ten pairs of circles.

289. Ten pairs of triangles are formed by six points on a conic; show that the middle points of the intersections of the perpendiculars, or centres of the nine-point circles, of the pairs of triangles lie on a line.

290. Four triangles are formed by four points on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\,;$$

show that the centres of the circles circumscribing these triangles lie on the conic

$$(4ax - c^2P)^2 + (4by + c^2Q)^2 = 4c^4 \sin^2 s,$$

where $\quad P = \Sigma \cos\phi, \quad Q = \Sigma \sin\phi, \quad 2s = \Sigma\phi,$

ϕ_1, &c., being the eccentric angles of the points on the curve.

291. To find the equation of the equilateral hyperbola passing through four points on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

If ϕ is the eccentric angle of any point x, y on the curve, we have

$$x = \frac{a(t^2+1)}{2t}, \qquad y = \frac{b(t^2-1)}{2t\sqrt{-1}},$$

putting $e^{\phi\sqrt{-1}} = t$. Substituting these values, then, in the equation of the equilateral hyperbola

$$x^2 - y^2 + 2hxy + 2gx + 2fy + c = 0,$$

we get

$$\{a^2+b^2-2hab\sqrt{-1}\}t^4+4\{ga-fb\sqrt{-1}\}t^3+2(2c+b^2-a^2)t^2$$
$$+4\{ga+fb\sqrt{-1}\}t+a^2+b^2+2hab\sqrt{-1}=0.$$

From the absolute term of this equation we obtain

$$2hab = (a^2 + b^2)\tan s,$$

where $$2s = \Sigma\phi\,;$$

and from the coefficients of t^3 and t we find

$$4ga = -\frac{(a^2 + b^2)}{\cos s}(P\cos s + Q\sin s),$$

$$4fb = -\frac{(a^2 + b^2)}{\cos s}(P\sin s - Q\cos s),$$

where $$P = \Sigma\cos\phi, \quad Q = \Sigma\sin\phi.$$

Also, from the coefficient of t^2, we have

$$2c = -(a^2 - b^2) + \frac{(a^2 + b^2)}{\cos s}R,$$

where $$R = \Sigma\cos\tfrac{1}{2}(\phi_1 + \phi_2 - \phi_3 - \phi_4)\,;$$

hence, finally, the equation of the equilateral hyperbola is

$$\begin{aligned}2ab\cos s\,(x^2 - y^2) + 2(a^2 + b^2)\sin s\,xy - b(a^2 + b^2)(P\cos s + Q\sin s)\,x \\ - a(a^2 + b^2)(P\sin s - Q\cos s)y - ab(a^2 - b^2)\cos s \\ + ab(a^2 + b^2)R = 0.\end{aligned}$$

292. Four tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

are drawn at the points whose eccentric angles are ϕ_1, &c.; show that the tangential equation of the parabola which touches these tangents is

$$\begin{aligned}a^2(R + \cos s)\lambda^2 + b^2(R - \cos s)\mu^2 + 2ab\sin s\,\lambda\mu \\ + a(P\cos s + Q\sin s)\nu\lambda + b(P\sin s - Q\cos s)\mu\nu = 0,\end{aligned}$$

where P, Q, R, s have the same meaning as in the preceding example.

293. A quadrilateral is formed by four tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\,;$$

show that the equation of the line passing through the middle points of the diagonals is

$$\frac{x}{a}(P \sin s - Q \cos s) - \frac{y}{b}(P \cos s + Q \sin s) = 0,$$

where P, Q, R, s have the same meaning as in Ex. 291.

294. Given five points, to show that there exists an equilateral hyperbola, such that the centre of the circle passing through any three of the points is the pole, with regard to the hyperbola, of the line bisecting at right angles the line joining the remaining two points.

Let

$$\rho_i^2 = (x - x_i)^2 + (y - y_i)^2,$$

then the equation

$$\Sigma_5^1 \, l_i \, \rho_i^4 = 0 \qquad (1)$$

will represent an equilateral hyperbola, provided we have

$$\Sigma l_i = 0, \quad \Sigma l_i x_i = 0, \quad \Sigma l_i y_i = 0, \quad \Sigma l_i (x_i^2 + y_i^2) = 0\,; \qquad (2)$$

for these are the relations which we obtain if we express that the coefficients of

$$(x^2 + y^2)^2, \quad x(x^2 + y^2), \quad y(x^2 + y^2),$$

and the sum of the coefficients of x^2 and y^2 in (1) vanish.

Now, when the relations (2) are satisfied, we have also

$$\Sigma_5^1 \, l_i \, \rho_i^2 = 0, \qquad (3)$$

and then it is easy to see that (1) can be written in the form

$$\Sigma_4^1 l_i \alpha_i^2 = 0, \qquad (4)$$

where $\alpha_1 = \rho_1^2 - \rho_5^2, \quad \alpha_2 = \rho_2^2 - \rho_5^2$, &c.;

and, from (3), $$\Sigma_4^1 l_i \alpha_i = 0. \qquad (5)$$

But when the curve is written in the form (4), we see from (5) that the pole of the line $\alpha_4 = 0$ is found from the equations

$$\alpha_1 = \alpha_2 = \alpha_3. \qquad (6)$$

Now the line $\alpha_4 = 0$ bisects at right angles the line joining the points $x_4 y_4$, $x_5 y_5$, and the equations (6) represent the centre of the circle passing through the points $x_1 y_1$, $x_2 y_2$, $x_3 y_3$; therefore, &c.

If the given points are taken on the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

the asymptotes of the hyperbola are parallel to the axes, and the co-ordinates of its centre are given by the equations

$$x = \frac{(a^2 - b^2)}{4a}(P + \cos 2s), \quad y = \frac{b^2 - a^2}{4b}(Q - \sin 2s),$$

where $P = \Sigma \cos \phi, \quad Q = \Sigma \sin \phi, \quad 2s = \Sigma\phi.$

295. Find the equations of the parabola and equilateral hyperbola having closest contact with a conic at a given point (see Ex. 291, and Ex. 292).

Also show that the locus of the centre of the hyperbola is the inverse of the curve with regard to its director circle.

296. A conic passes through four fixed points on a circle; to find the locus of its vertices.

Let $$V \equiv x^2 + y^2 - k^2 = 0$$

be the equation of the circle, and

$$U \equiv ax^2 + by^2 + 2gx + 2fy = 0$$

that of one of the conics passing through the points; then

$$U + \lambda V = 0 \qquad (1)$$

represents any conic of the system. Now the equations

$$ax + g + \lambda x = 0, \qquad (2)$$

$$by + f + \lambda y = 0, \qquad (3)$$

represent respectively the axes of (1); hence, eliminating λ between (1) and (2) and (1) and (3), we get

$$(a-b)\,xy^2 + g(y^2 - x^2) - 2fxy - k^2(ax+g) = 0, \qquad (4)$$

$$(a-b)\,x^2y + f(y^2 - x^2) + 2gxy + k^2(by+f) = 0. \qquad (5)$$

These two cubics evidently pass through the four points, and also through the intersections of the diagonals and opposite sides of the quadrilateral formed by these points; for it is easy to see that the equations (4) and (5) are satisfied, if we have

$$\frac{ax+g}{x} = \frac{by+f}{y} = -\frac{(gx+fy)}{k^2};$$

but these are the equations which determine the common self-conjugate triangle of U and V; therefore, &c.

It may be observed that the cubics (4) and (5) cut each other orthogonally at their seven finite points of intersection which we have found above.

297. Two conics U and V are inscribed in the same quad-

rilateral so that a focus of U coincides with the centre of V; to show that the points of contact of V with the sides lie on a circle.

Let
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

be the equation of V, then

$$\nu(x'\lambda+y'\mu+\nu)-\beta^2(\lambda^2+\mu^2)=0$$

is the tangential equation of U; for this represents a conic, of which the origin and the point x', y' are foci. Now if we express that the tangent of V represented by

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0 \qquad (1)$$

touches U, we obtain

$$1-\frac{x'}{a}\cos\theta-\frac{y'}{b}\sin\theta-\frac{\beta^2}{a^2b^2}(a^2\sin^2\theta+b^2\cos^2\theta)=0. \quad (2)$$

But this equation (2) shows that the point of contact $a\cos\theta$, $b\sin\theta$ of (1) lies on

$$1-\frac{x'x}{a^2}-\frac{y'y}{b^2}-\frac{\beta^2}{a^2b^2}(a^2+b^2-x^2-y^2)=0,$$

which represents a circle passing through the points where the polar of the second focus of U meets the director circle of V.

298. If the two conics described through a point P to touch four fixed lines cut orthogonally at P, to show that P lies on the circular cubic which is the locus of the foci of the conics touching the lines.

If from any point P tangents are drawn to the system of conics inscribed in a quadrilateral, they will belong to a system in involution of which the tangents to the conics of the system which pass through P are double lines (*Conics*, Art. 344); but since, in the case we are considering, the latter pair of lines are at right angles to one another, it follows that the tangents from P to one conic of the system must pass through the circular points at infinity, or, in other words, P must be a focus of that conic; therefore, &c.

In a similar manner we can show that the two parabolæ described through any point P of the circumscribing circle to touch the sides of a triangle cut each other orthogonally at P.

299. Two conics are described through a point P to touch the lines A, B, C, D, and two more to touch the lines A, B, C, E. If the tangents to these four conics at P have a constant anharmonic ratio, show that the locus of P is a conic touching D and E.

Also show that, if the ratio becomes a harmonic one, the locus will reduce to a right line.

300. A tangent to the conic

$$S \equiv ax^2 + by^2 + 2gx + 2fy + c = 0$$

is at right angles to a tangent to the conic

$$S' \equiv bx^2 + ay^2 - 1 = 0,$$

the axes being rectangular; to find the locus of their intersection.

If θ is the angle which a tangent from xy to S makes with the axis of x, we obtain from the equation of the pair of tangents through xy (*Conics*, Art. 92),

$$S(a \cos^2\theta + b \sin^2\theta) = \{(ax+g)\cos\theta + (by+f)\sin\theta\}^2; \quad (1)$$

and similarly for S' we have

$$S'(b\cos^2\theta' + a\sin^2\theta') = (bx\cos\theta' + ay\sin\theta')^2; \qquad (2)$$

but $\theta' - \theta = \frac{1}{2}\pi$; hence (2) becomes

$$S'(a\cos^2\theta + b\sin^2\theta) = (bx\sin\theta - ay\cos\theta)^2. \qquad (3)$$

We have then from (1) and (3), putting $\frac{S}{S'} = \mu^2$,

$$\tan\theta = \frac{\mu ay + ax + g}{\mu bx - by - f}.$$

Substituting this value of θ in (3), we get

$$S'\{a(\mu bx - by - f)^2 + b(\mu ay + ax + g)^2\} = \{ab(x^2+y^2) + bgx + afy\}^2;$$

or, restoring the value of μ,

$$abS(1 + S') + S'(abS + af^2 + bg^2 - abc) +$$

$$2ab\sqrt{SS'}\{(a-b)xy + gy - fx\} = (ab(x^2+y^2) + bgx + afy)^2. \quad (4)$$

Now we can show that we have identically

$$\{ab(x^2+y^2) + bgx + afy\}^2 = (1 + S')(abS + af^2 + bg^2 - abc)$$
$$- ab\{(a-b)xy + gy - fx\}^2;$$

hence (4) becomes

$$\{(a-b)xy + gy - fx\}^2 + 2\sqrt{SS'}\{(a-b)xy + gy - fx) + SS'$$
$$= \frac{af^2 + bg^2 - abc}{ab}.$$

We have therefore, finally,

$$\left\{(a-b)xy + gy - fx \pm \sqrt{\left(\frac{f^2}{b} + \frac{g^2}{a} - c\right)}\right\}^2 = SS', \qquad (5)$$

which represents a pair of curves of the fourth order, each having quartic contact with the given conics.

It may be shown that the curves (5) are bicircular quartics with nodes.

301. In the same way as in the preceding example, we can show that if a tangent to the parabola $y^2 + ax + c = 0$ is at right angles to a tangent to the parabola $x^2 + by + c' = 0$, then their intersection will lie on

$$(2abxy - ab^2x - a^2by - cb^2 - c'a^2)^2 - 4a^2b^2SS' = 0,$$

which represents a nodal circular cubic having triple contact with the two parabolæ.

302. A tangent to the circle

$$S \equiv y^2 + (x - c)^2 - r^2 = 0$$

is inclined at a constant angle θ to a tangent to the circle

$$S' \equiv y^2 + (x + c)^2 - r'^2 = 0\,;$$

show that their intersection lies on one or other of the curves

$$\{\cos\theta\,(x^2 + y^2 - c^2) + 2cy\sin\theta \pm rr'\}^2 - SS' = 0.$$

Show also that these curves are limaçons of Pascal.

303. If the tangents drawn to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

from xy form a harmonic pencil with the perpendiculars to the tangents to the conic

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0,$$

from the same point, show that the locus of xy is the bicircular quartic

$$(x^2 + y^2)^2 - (a^2 + a'^2)\,x^2 - (b^2 + b'^2)\,y^2 + a^2a'^2 + b^2b'^2 = 0.$$

304. If the tangents from a point P to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \qquad (1)$$

contain an angle α; to show that the tangents from P to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{4a^2b^2 + (a^2 - b^2)^2 \sin^2 \alpha}{(a^4 - b^4) \sin^2 \alpha} \qquad (2)$$

contain an angle β, where

$$\cos \beta = \left(\frac{a^2 - b^2}{a^2 + b^2}\right) \cos \alpha. \qquad (3)$$

This is evidently the case if the loci of intersections of tangents to the conics at the constant angles α and β, respectively, coincide. These loci are for the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \qquad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0$$

(*Conics*, Art. 169, Ex. 3),

$$(x^2 + y^2 - a^2 - b^2)^2 - 4 \cot^2 \alpha \, (b^2 x^2 + a^2 y^2 - a^2 b^2) = 0,$$

$$(x^2 + y^2 - a'^2 - b'^2)^2 - 4 \cot^2 \beta \, (b'^2 x^2 + a'^2 y^2 - a'^2 b'^2) = 0.$$

But, expressing that these two curves coincide, we obtain the conics (1) and (2) given above, and the relation (3) between the angles.

305. A tangent to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

is inclined at a constant angle θ to a tangent of the confocal conic

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0;$$

show that the locus of their intersection is the bicircular quartic

$$\tan^2\theta\,(x^2+y^2-a^2-b'^2)^2-2\,(b^2+b'^2)\,x^2-2\,(a^2+a'^2)\,y^2$$
$$+2\,(a^2b^2+a'^2b'^2)+(a^2-a'^2)^2\cot^2\theta=0.$$

306. A, B, A', B' are the points of contact of the tangents drawn from a point P to two conics having the same centre and axes; if the circles passing through P, A, B, and P, A', B', respectively, touch one another, show that the locus of P is a bicircular quartic.

307. The pairs of tangents drawn from a point P to two concentric equilateral hyperbolæ contain equal angles; show that the locus of P is an equilateral hyperbola concentric with the given ones, and passing through their intersection.

308. Let x, y, be the co-ordinates of the vertex C of a triangle self-conjugate with regard to the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0;$$

then if the angle at C is right, show that the co-ordinates x', y' of the middle points of the base are given by the equations

$$2x'=\frac{a^2}{c^2x}(x^2+y^2+c^2),\qquad 2y'=-\frac{b^2}{c^2y}(x^2+y^2-c^2).$$

309. A point P moves along a right line; show that the locus of the foot of the perpendicular from P on its polar, with regard to a conic, is a circular cubic passing through the foci of the conic.

310. S, H are the given foci of a conic U, and S', H' of a conic V; if U and V vary so as to be always similar to

each other, show that their common tangents envelop a conic which touches the lines SS', SH', $S'H$, HH'.

311. The perpendicular to a chord of a conic at its middle point passes through a fixed point O on one of the axes of the curve; show that the chord touches a parabola, of which O is the focus.

312. From any point of the circle

$$x^2 + y^2 - a^2 - b'^2 = 0,$$

pairs of tangents are drawn to the confocal conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \qquad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} - 1 = 0;$$

show that the difference of the squares of the reciprocals of their lengths is the same for each pair.

313. If from any point of the quartic curve

$$2(x^2 + y^2)\left(1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}\right) - (a^2 - b^2)\left(\frac{x^2}{a^3} - \frac{y^2}{b^2}\right) = 0$$

tangents are drawn to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

show that the sum of the squares of their lengths is equal to $a^2 - b^2$.

314. If the foot of the perpendicular from a point P on its polar, with regard to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

moves along the right line $\lambda x + \mu y + \nu = 0$, show that the locus of P is the cubic

$$\lambda b^2 x + \mu a^2 y + (a^2 - b^2)\,xy\left(\frac{\lambda x}{b^2} - \frac{\mu y}{a^2}\right) + \nu a^2 b^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) = 0.$$

315. Two tangents to a conic, whose foci are given, pass through fixed points on the axis minor; show that the locus of their intersection is a circle passing through the foci. If the tangents, instead of passing through fixed points, are parallel to fixed lines, show that the locus is an equilateral hyperbola passing through the foci.

316. Given two parabolæ with their axes parallel, show that a right line which cuts off from them areas, which are in a constant ratio to each other, envelops a parabola which touches the common tangents of the given ones.

317. A variable tangent to a conic S, whose point of contact is P, meets a concentric, similar, and similarly situated conic in A, B; show that the lines joining A, B to the points where the normal to S at P meets the axes of S are inclined to A, B at constant angles.

318. A conic passes through four fixed points; to find the envelope of the right line which passes through the middle points of the diagonals of the quadrilateral formed by drawing the tangents at these points.

If we consider two consecutive curves of the system, we see that the different loci of the centre, obtained according as the tangents or points are fixed, must touch each other. Hence, the line referred to above, which is the locus when the tangents are fixed, must touch the locus of the centre of the system. The envelope is, therefore, the conic bisecting all the lines joining the points.

319. To show, in the preceding example, that the line containing the intersections of the perpendiculars of the four triangles formed by the tangents passes through a fixed point.

This line is the radical axis of the director circles of conics touching four lines (*Conics*, Art. 298, Ex. 1), and is, therefore, the chord of contact of the director circle of the system of conics. But the chord of contact of the director circle, given four points on a conic, passes through a fixed point, viz., the centre of the circle circumscribing the fixed self-conjugate triangle (see Ex. 286).

320. Conics of a given system are described to have double contact with two fixed conics; show that (1) the line passing through the middle points of the diagonals of the quadrilateral formed by the tangents at the points of contact touches a fixed conic; (2) the line of the intersections of the perpendiculars of the four triangles formed by the same tangents passes through a fixed point.

321. Show that there are a real pair of lines passing through the points of intersection of the point circle

$$\rho^2 \equiv (x - x')^2 + (y - y')^2 = 0,$$

and the conic $$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

whose equations are

$$L \equiv \frac{\nu x}{a} + \sqrt{(c^2 - \nu^2)}\,\frac{y}{b} - \frac{1}{c}\{a\mu + b\sqrt{(\mu^2 - c^2)}\} = 0,$$

$$M \equiv \frac{\nu x}{a} - \sqrt{(c^2 - \nu^2)}\,\frac{y}{b} - \frac{1}{c}\{a\mu - b\sqrt{(\mu^2 - c^2)}\} = 0,$$

where μ, ν are the elliptic co-ordinates of x', y' (see Ex. 1).

Show also that we have the identity

$$\rho^2 = (a^2 - \nu^2)\,S + LM.$$

322. If

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{b}\sin\tfrac{1}{2}(\alpha + \beta) - \cos\tfrac{1}{2}(\alpha - \beta) = 0$$

is the equation of a chord of the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0\,;$$

show that the elliptic co-ordinates of the antipoints (Salmon's *Higher Plane Curves*, Art. 139) of its extremities are given by the equations

$$\mu = a \cos \tfrac{1}{2}(\alpha - \beta) \pm b\sqrt{-1}\,\sin \tfrac{1}{2}(\alpha - \beta), \quad \nu = c \cos \tfrac{1}{2}(\alpha + \beta).$$

323. Given five lines touching the parallel curve to a conic; to find the locus of the centre of the curve.

Using the notation of *Conics*, Art. 228, Ex. 8, we have

$$(\alpha - r)^2 = a^2 \cos^2(\theta - \alpha) + b^2 \sin^2(\theta - \alpha),$$

and similar equations for β, &c., where r is the constant distance of the parallel curve. Hence, eliminating linearly r

$$a^2\cos^2\theta + b^2\sin^2\theta - r^2, \quad (a^2 - b^2)\sin\theta\cos\theta, \quad a^2\sin^2\theta + b^2\cos^2\theta - r^2$$

from five such equations, we obtain a determinant which may be written

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \delta, & \epsilon \\ \alpha^2, & \beta^2, & \gamma^2, & \delta^2, & \epsilon^2 \\ \cos 2\alpha, & \cos 2\beta, & \cos 2\gamma, & \cos 2\delta, & \cos 2\epsilon \\ \sin 2\alpha, & \sin 2\beta, & \sin 2\gamma, & \sin 2\delta, & \sin 2\epsilon \\ 1, & 1, & 1, & 1, & 1 \end{vmatrix} = 0.$$

If we substitute for α, β, &c.,

$$x\cos\alpha + y\sin\alpha - p_1, \quad x\cos\beta + y\sin\beta - p_2, \text{ \&c.},$$

we can show that terms of higher degree than the second vanish in this determinant. The locus is, therefore, a conic. Since r may have a double sign, we can change the signs of α, β, &c., in the determinant. We thus see that there are sixteen such conics altogether.

324. A conic passes through four fixed points; show that the locus of the pole of a fixed triangle with regard to the conic (*Conics*, Art. 375) is a unicursal quartic, of which the vertices of the triangle are nodes.

325. A conic touches four fixed lines; show that the locus of the pole of a fixed triangle, with respect to the conic, is a conic circumscribing the fixed triangle.

326. Given four tangents to the curve parallel to a parabola; show that the locus of the focus of the parabola consists of eight nodal circular cubics. Show also that each of these cubics passes through the centre of one of the circles touching the sides of a triangle formed by any three of the tangents.

327. Given four lines parallel to the tangents to a conic at the constant distance $\sqrt{\left(\frac{a^2+b^2}{2}\right)}$, where a, b are the semi-axes of the curve; show that the locus of the centre consists of eight circular cubics.

328. $ABCD$ is a fixed parallelogram circumscribed about a conic; if any tangent to the curve meet AB, CD in the points P, Q, respectively, show that the area of the triangle PAQ is constant.

329. P, P' are the points of contact of a common tangent of two conics; if C is the centre of one of the conics, and A the area of the triangle CPP', show that, taking the four common tangents,

$$\Sigma \frac{1}{A} = 0.$$

330. Show that the polar equation of the evolute of a parabola referred to the focus may be written in the form

$$\left(\frac{m}{\rho}\right)^{\frac{1}{3}} = (\cos \tfrac{1}{2}\theta)^{\frac{2}{3}} - (\sin \tfrac{1}{2}\theta)^{\frac{2}{3}}.$$

If the evolute cut a parabola having the same focus and axis at an angle ϕ, show that

$$\cot^3 \phi = \cot \tfrac{1}{2}\theta.$$

331. A tangent to a hyperbola U meets the asymptotes in A, B; if a circle S is described through A, B, so that AB subtends a constant angle at the circumference, show that the locus of the centre of S is a conic having the same centre and axes as U.

Also show that S cuts a fixed circle orthogonally.

332. If we substitute a concentric, similar, and similarly situated conic for the asymptotes in the preceding example, show that the locus of the centre of S is a concentric conic.

XV.—SPHERO-CONICS.

333. If we take the principal axes of the cone of the second order containing a sphero-conic as axes of co-ordinates, we may consider the curve as defined by the equations

$$ax^2 + by^2 + cz^2 = 0, \qquad (1)$$

$$x^2 + y^2 + z^2 - 1 = 0, \qquad (2)$$

the radius of the sphere being taken equal to unity.

Let α, β be the halves of the principal angles of the curve, then (1) becomes

$$x^2 \cot^2 \alpha + y^2 \cot^2 \beta - z^2 = 0 \qquad (3)$$

if the axis of z contains the internal centre of the curve. We have also, from (2) and (3),

$$\frac{x^2}{\sin^2 \alpha} + \frac{y^2}{\sin^2 \beta} - 1 = 0. \qquad (4)$$

When the curve is written in the forms (3) and (4), the real foci are given by

$$x = \pm \sin \gamma, \qquad y = 0, \qquad z = \cos \gamma,$$

where
$$\cos \gamma = \frac{\cos \alpha}{\cos \beta};$$

and the equation of the real cyclic arcs is

$$y^2 (\sin^2 \alpha - \sin^2 \beta) - z^2 \sin^2 \beta = 0.$$

We may conveniently express the co-ordinates of any point on the curve in terms of a single parameter, as follows:—

$$x = \frac{\sin a \cos \theta}{\Delta(\theta)}, \quad y = \frac{\tan\beta \cos a \sin\theta}{\Delta(\theta)}, \quad z = \frac{\cos a}{\Delta(\theta)}, \quad (5)$$

where $$\Delta(\theta) = \surd(1 - \sin^2\gamma \sin^2\theta);$$

or thus: $x = \sin a \cos\phi,\ y = \sin\beta \sin\phi,\ z = \cos\beta\Delta(\phi),$ (6)

where $$\Delta(\phi) = \surd(1 - \sin^2\gamma \cos^2\phi).$$

334. To find the equation of the circle passing through three points on the curve.

Let
$$U \equiv ax^2 + by^2\ cz^2 = 0$$

be the cone containing the curve, and

$$V \equiv (lx + my + nz)^2 - (x^2 + y^2 + z^2) = 0$$

that containing the circle; then if we form the discriminant of $U - kV$, we obtain

$$\frac{l^2}{a+k} + \frac{m^2}{b+k} + \frac{n^2}{c+k} - \frac{1}{k} = 0. \qquad (1)$$

But writing $U - kV$ in the form

$$(a+k)x^2 + (b+k)y^2 + (c+k)z^2 - k(lx + my + nz)^2,$$

we see that when it breaks up into two great circles, they must be tangents to the conic

$$(a+k)x^2 + (b+k)y^2 + (c+k)z^2 = 0. \qquad (2)$$

Now, if we use the angle θ at (5), Ex. 333, the equation of the chord joining the points θ_1, θ_2 can be written

$$x\sqrt{a}\cos\tfrac{1}{2}(\theta_1+\theta_2)+y\sqrt{b}\sin\tfrac{1}{2}(\theta_1+\theta_2)$$
$$=z\sqrt{-c}\cos\tfrac{1}{2}(\theta_1-\theta_2),$$

and if this chord touch the conic (2), we have

$$\frac{a\cos^2\frac{1}{2}(\theta_1+\theta_2)}{a+k}+\frac{b\sin^2\frac{1}{2}(\theta_1+\theta_2)}{b+k}=\frac{c\cos^2\frac{1}{2}(\theta_1-\theta_2)}{c+k}.$$

But comparing this equation with

$$(\cos\sigma-\Delta)\cos^2\tfrac{1}{2}(\theta_1+\theta_2)+(1+\cos\sigma)\sin^2\tfrac{1}{2}(\theta_1+\theta_2)$$
$$=(1-\Delta)\cos^2\tfrac{1}{2}(\theta_1-\theta_2), \qquad (3)$$

where $$\Delta=\sqrt{(1-\lambda^2\sin^2\sigma)},$$

we get $$\frac{\cos\sigma-\Delta}{1-\Delta}=\frac{a(c+k)}{c(a+k)},\quad \frac{1+\cos\sigma}{1-\Delta}=\frac{b(k+c)}{c(k+b)};$$

whence $$\frac{1-\Delta^2}{1-\cos^2\sigma}=\lambda^2=\frac{c(a-b)}{b(a-c)}, \qquad (4)$$

and $$\left.\begin{aligned}\frac{1-\cos\sigma}{1+\Delta}&=\left(\frac{a-c}{a-b}\right)\left(\frac{k+b}{k+c}\right)\\ \frac{\Delta+\cos\sigma}{1+\Delta}&=\left(\frac{c-b}{a-b}\right)\left(\frac{k+a}{k+c}\right)\end{aligned}\right\}. \qquad (5)$$

Now, from (3), we have

$$F(\theta_1)+F(\theta_2)=F(\sigma),$$

and, from (5), $$F(\sigma)=2F(\phi),$$

if $$\cos^2\phi=\left(\frac{c-b}{a-b}\right)\left(\frac{k+a}{k+c}\right), \qquad (6)$$

in which case we have also

$$\sin^2\phi = \left(\frac{a-c}{a-b}\right)\left(\frac{k+b}{k+c}\right),\ 1 - \lambda^2\sin^2\phi = \frac{(b-c)\,k}{b\,(k+c)}. \qquad (7)$$

But from the absolute term of the equation (1) in k, we have

$$l^2 + m^2 + n^2 - 1 = \frac{a\,b\,c}{k_1\,k_2\,k_3}\,; \qquad (8)$$

and from the absolute term of the equation in $k + a$ we get

$$\frac{a(a-b)(a-c)l^2}{l^2+m^2+n^2-1} = (k_1+a)(k_2+a)(k_3+a), \qquad (9)$$

and similar values for m^2, n^2; hence we may write the equation of the plane of the circle thus:

$$x\sqrt{\left\{\frac{(a+k_1)(a+k_2)(a+k_3)}{a\,(a-b)(a-c)}\right\}} + \&\text{c.} = \sqrt{\left(\frac{k_1\,k_2\,k_3}{a\,b\,c}\right)}. \quad (10)$$

Putting, now, $a = \cot^2\alpha,\ b = \cot^2\beta,\ c = -1,$

we have $$F(\theta_1) = \int_0^{\theta_1}\frac{d\theta_1}{\sqrt{(1-\sin^2\gamma\sin^2\theta_1)}} = u_1, \text{ say;}$$

and then we get

$$\frac{k+a}{k+c} = \left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)\text{cn}^2\tfrac{1}{2}(u_1+u_2),$$

$$\frac{k+b}{k+c} = -\frac{(\sin^2\alpha - \sin^2\beta)}{\sin^2\beta}\,\text{sn}^2\tfrac{1}{2}(u_1+u_2),$$

$$\frac{k}{k+c} = \cos^2\beta\,\text{dn}^2\tfrac{1}{2}(u_1+u_2).$$

Thus (10) becomes

$$x\,\frac{(\sin^2\alpha - \sin^2\beta)}{\sin\alpha\,\cos\alpha}\,c_1 c_2 c_3 + y\,\frac{(\sin^2\beta - \sin^2\alpha)}{\sin\beta\,\cos\beta}\,s_1 s_2 s_3 + z = \frac{\cos^2\beta}{\cos\alpha}\,d_1 d_2 d_3, \qquad (11)$$

where $c_1 = \operatorname{cn}\frac{1}{2}(u_2 + u_3)$, &c., $s_1 = \operatorname{sn}\frac{1}{2}(u_2 + u_3)$, &c.,

and $d_1 = \operatorname{dn}\frac{1}{2}(u_2 + u_3)$, &c.

We have also

$$k_1 = -\frac{-\cos^2\beta\,\operatorname{dn}^2\frac{1}{2}(u_2 + u_3)}{\sin^2\alpha\,\operatorname{sn}^2\frac{1}{2}(u_2 + u_3) + \sin^2\beta\,\operatorname{cn}^2\frac{1}{2}(u_2 + u_3)}, \qquad (12)$$

and similar values for k_2, k_3. Now if R is the spherical radius of the circle, we find, from (8),

$$\tan^2 R = \frac{abc}{k_1 k_2 k_3};$$

hence, from (12), we have an expression for R in terms of u_1, u_2, u_3.

335. From (2) in the preceding example we see that when k, and therefore $u_1 + u_2$, is given, the chord touches a concyclic conic. In this case the point of contact is at the external point of bisection of the chord. If $u_1 - u_2$ is given, the chord touches a concyclic conic, and the point of contact is then the internal point of bisection (see Salmon's *Surfaces*, Art. 247).

Hence we see that if two sides of an inscribed triangle touch two fixed concyclic conics externally, then the third side will touch a fixed concyclic conic internally; and from (11), in the preceding example, the centre of the circumscribing circle will lie on a fixed sphero-conic, having the same centres as the given one.

If we seek the locus of the centre of the circle which lies in its plane, we obtain an equation of the form

$$(x^2 + y^2 + z^2)^2 - (\alpha x^2 + \beta y^2 + \gamma z^2) = 0.$$

This equation, which represents the pedal surface of a quadric, being combined with the cone of the second order found above, gives the required locus.

336. To find the locus of the intersection of the perpendiculars of triangles inscribed in one sphero-conic, and circumscribed about another.

The equation (1), Ex. 334, gives the discriminant of $U - kV$, where U is a sphero-conic and V a circle.

Putting

$$l = x' \sec\rho, \quad m = y' \sec\rho, \quad n = z' \sec\rho$$

in this equation, where x', y', z' are the co-ordinates of the centre of V and ρ is its radius, it becomes

$$k^3 \sin^2\rho + k^2\{(b+c)\,x'^2 + (c+a)\,y'^2 + (a+b)\,z'^2 - (a+b+c)\cos^2\rho\}$$
$$+ k\{bcx'^2 + cay'^2 + abz'^2 - (ab+bc+ca)\cos^2\rho\}$$
$$- abc\cos^2\rho = 0. \quad (1)$$

Now, if V is the polar circle of a triangle inscribed in U, the coefficient of k^2 vanishes; and if V is the polar circle of a triangle circumscribed about U, the coefficient of k vanishes. Hence, if U and U' are the given conics, by equating the values of ρ we obtain the equation of the locus

$$\frac{ax^2 + by^2 + cz^2}{a+b+c} = \frac{a'(b'+c')\,X^2 + b'(c'+a')\,Y^2 + c'(a'+b')\,Z^2}{a'b' + b'c' + c'a'},$$

where x, y, z are the axes of U, and X, Y, Z those of U'.

337. To find the locus of the centre of the circumscribing circle of a triangle inscribed in one sphero-conic and circumscribed about another.

Using the notation of the preceding example, if U is the conic about which the triangle is circumscribed, we must have

$$\Theta'^2 - 4\Delta'\Theta = 0, \quad (1)$$

where we write (1), Ex. 336, in the form

$$\Delta k^3 + \Theta k^2 + \Theta' k + \Delta'.$$

Now, exactly as in the case of plane conics (see Ex. 35), we can show that the triangle is self-conjugate with regard to a fixed conic U'; thus we have

$$\frac{b'c'X^2 + c'a'Y^2 + a'b'Z^2}{a'b' + b'c' + c'a'} = \cos^2\rho, \quad (2)$$

where $$a'X^2 + b'Y^2 + c'Z^2 = 0$$

is the equation of U'. Hence, eliminating ρ between (1) and (2), we get

$$(W - W')^2 = 4abcW'\{(a+b+c)W' - (b+c)x^2 - (c+a)y^2 - (a+b)z^2\},$$

where $$W = \frac{bcx^2 + cay^2 + abz^2}{bc + ca + ab},$$

$$W' = \frac{b'c'X^2 + c'a'Y^2 + a'b'Z^2}{b'c' + c'a' + a'b'}.$$

338. To find the locus of the centres of equilateral triangles inscribed in a sphero-conic.

Let P be the radius of the imaginary polar circle of the triangle, then we have, as in plano,

$$U = ax^2 + by^2 + cz^2 = (a+b+c)\sin^2 P. \quad (1)$$

But if r is the radius of the inscribed circle, we have, from the invariant relation connecting the curve with the inscribed circle of an inscribed triangle,

$$\{U-(a+b+c)\sin^2 r\}^2 = 4\sin^2 r\{W-(ab+bc+ca)\cos^2 r\}, \quad (2)$$

where $$W = bcx^2 + cay^2 + abz^2.$$

We now want to find the relation connecting P and r for an equilateral triangle. The relation for any triangle is

$$\sec^2 P \sec^2 r \sin^2 D = \tan^2 P + 2\tan^2 r$$

(see Salmon's *Surfaces*, Art. 257). But for an equilateral triangle $D = 0$; therefore

$$\tan^2 r = -\tfrac{1}{2}\tan^2 P\,;$$

hence, from (1) and (2), we get

$$9U(U-p\Omega)^2 - 4\Omega UW - 8\Omega(U-p\Omega)(W-q\Omega) = 0,$$

where $$p = a+b+c, \quad q = ab+bc+ca,$$

$$\Omega = x^2+y^2+z^2.$$

339. To find the equation of the circle inscribed in a triangle circumscribed about a sphero-conic.

We can arrive at the equation of the circle by a method similar to that we employed in Ex. 44.

Let

$$\left.\begin{aligned} \frac{x^2}{\mu^2} + \frac{y^2}{\mu^2-b^2} - \frac{z^2}{c^2-\mu^2} &= 0 \\ \frac{x^2}{\nu^2} - \frac{y^2}{b^2-\nu^2} - \frac{z^2}{c^2-\nu^2} &= 0 \end{aligned}\right\} \quad (1)$$

be the equations of two intersecting confocal cones of the second order; then, if x', y', z' are the co-ordinates of the centre of the circle, we find

$$\left.\begin{aligned} x' &= \frac{\nu_1\nu_2\nu_3}{\sin\alpha\sin\gamma},\quad y' = \frac{\sqrt{\{(b^2-\nu_1^2)(b^2-\nu_2^2)(b^2-\nu_3^2)\}}}{\sin\gamma\cos\gamma\sqrt{(\sin^2\alpha-\sin^2\gamma)}}, \\ z' &= \frac{1}{\cos\alpha\cos\gamma}\sqrt{\{(c^2-\nu_1^2)(c^2-\nu_2^2)(c^2-\nu_3^2)\}}, \end{aligned}\right\} \quad (2)$$

where $\sin\gamma = \frac{b}{c}$ and $\frac{\nu_1}{c}$, $\frac{\nu_2}{c}$, $\frac{\nu_3}{c}$

are the sines of half the greatest axes of the confocal sphero-conics which pass through the vertices of the triangle. For the inscribed circle all these confocal sphero-conics intersect the given curve in real points; and for an exscribed circle, one of the confocals intersects the curve in real points.

If r is the radius of the inscribed circle, we find

$$\tan r = \frac{\sqrt{\{(a^2-\nu_1^2)(a^2-\nu_2^2)(a^2-\nu_3^2)\}}}{a\sqrt{\{(a^2-b^2)(a^2-c^2)\}}}, \quad (3)$$

where a is the value of μ for the given curve. Again, if s is the semiperimeter, we find

$$\sin s = \frac{\sqrt{\{(\mu_1^2-a^2)(\mu_2^2-a^2)(\mu_3^2-a^2)\}}}{a\sqrt{\{(a^2-b^2)(a^2-c^2)\}}}. \quad (4)$$

If we substitute μ_2, μ_3 for ν_2, ν_3 in (2), (3), and (4), we obtain the corresponding formulæ for one of the circles exscribed to the triangle.

340. Triangles are inscribed in one sphero-conic, and circumscribed about another; show that the centres of the circles touching the sides lie on the intersection of the sphere with a cone of the fourth order (see Ex. 337).

341. To find the locus of the centres of equilateral triangles circumscribed about a sphero-conic.

Let P and R be the radii of the polar and circumscribing circles of the triangle, respectively; then, from the invariant relation connecting the curve with the circumscribing circle of a circumscribing triangle, we have

$$(W - q\cos^2 R)^2 = 4abc\cos^2 R\,(U - p\sin^2 R); \qquad (1)$$

and from the relation connecting the curve with the polar circle of the same triangle, we have

$$W = q\cos^2 P, \qquad (2)$$

where U and W have the same meaning as at Ex. 338. But

$$\tan^2 P = -\tfrac{1}{2}\tan^2 R$$

for an equilateral triangle; hence, from (1) and (2), we get

$$9W(W - q\Omega)^2 - 4abc\,\Omega UW - 8abc\Omega\,(U - p\Omega)(W - q\Omega) = 0.$$

342. To find the locus of the centres of equilateral triangles self conjugate with regard to a sphero-conic.

With the notation of Ex. 338, we have, from the invariants of the curve and a circle,

$$W = q\cos^2 R, \quad U = P\sin^2 r;$$

but $\tan R = 2\tan r$ for an equilateral triangle; hence we get

$$3UW + \Omega\,(pW + qU) - pq\Omega^2 = 0.$$

343. A triangle is self-conjugate with regard to a sphero-conic; show that the feet of the perpendiculars form a triangle circumscribed about a confocal sphero-conic.

344. A circle S touches the sides of a triangle self-conjugate with regard to a sphero-conic U; show that the centre

of S lies on the equilateral sphero-conic having double contact with U at a pair of points which lie on a tangent to S.

An equilateral sphero-conic is such that the intersection of the perpendiculars of an inscribed triangle, and the centres of the circles touching the sides of a self-conjugate triangle, lie on the curve; that is, when the curve referred to its axes is written in the form

$$ax^2 + by^2 + cz^2 = 0,$$

we have

$$a + b + c = 0.$$

345. To find the equation of the polar circle of a triangle formed by two tangents and their chord of contact.

Let $x'y'z'$ be the co-ordinates of the vertex, and $x_1y_1z_1$, $x_2y_2z_2$ those of the points of contact of the sides, then if λ, μ, ν are the tangential co-ordinates of a tangent of the curve, we must have an equation of the form

$$(\lambda x_1 + \mu y_1 + \nu z_1)(\lambda x_2 + \mu y_2 + \nu z_2) - k(\lambda x' + \mu y' + \nu z')^2 \equiv \theta(bc\lambda^2 + ca\mu^2 + ab\nu^2); \quad (1)$$

but since the sum of the coefficients of x^2, y^2, and z^2 must be the same on both sides of this equation, we get

$$\cos\theta - k = \theta(bc + ca + ab), \qquad (2)$$

where θ is the length of the base.

Now λ, μ, ν are the co-ordinates of a point on the reciprocal curve

$$\Sigma\ (\equiv bcx^2 + cay^2 + abz^2);$$

hence, from (1) and (2), we get the identity

$$PQ - \cos\theta\Omega \equiv k\{\Omega^2 - (xx' + yy' + zz')^2\} + \frac{(\cos\theta - k)}{ab + bc + ca}\{a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2\}, \quad (3)$$

where $$\Omega = x^2 + y^2 + z^2.$$

Now $$PQ - \cos\theta\Omega \qquad (4)$$

is the sphero-conic which is the locus of the vertices of right-angled triangles described on the base. It is not difficult, then, to see from (3) that the polar of the vertex with regard to this conic coincides with the polar of the same point with regard to the conic

$$a(b+c)x^2 + b(c+a)y^2 + c(a+b)z^2 = 0. \qquad (5)$$

Again, it is easy to see that the vertex and the intersection of the perpendiculars are conjugate with regard to the locus (4), and, therefore, also with regard to (5); hence the intersection of the perpendiculars lies on the great circle

$$a(b+c)xx' + b(c+a)yy' + c(a+b)zz' = 0; \qquad (6)$$

and since it also lies on the perpendicular from the vertex on its polar with regard to the curve, we have

$$(b-c)y'z'x + (c-a)z'x'y + (a-b)x'y'z = 0. \qquad (7)$$

We thus determine the co-ordinates of the centre of the circle, and the remaining condition may then be obtained by expressing that the vertex and base are pole and polar with regard to the circle. We have, then, finally, if the equation of the circle is

$$(\lambda x + \mu y + \nu z)^2 - \rho^2(x^2 + y^2 + z^2) = 0,$$

$$\lambda = x'\{b(c+a)(a-b)y'^2 - c(a+b)(c-a)z'^2\}, \qquad (8)$$

$$\mu = y'\{c(a+b)(b-c)z'^2 - a(b+c)(a-b)x'^2\}, \qquad (9)$$

$$\nu = z'\{a(b+c)(c-a)x'^2 - b(c+a)(b-c)y'^2\}, \qquad (10)$$

$$\rho^2 = \{a^2(b+c)x'^2 + b^2(c+a)y'^2 + c^2(a+b)z'^2\}$$
$$\times\{a(b-c)^2y'^2z'^2 + b(c-a)^2z'^2x'^2 + c(a-b)^2x'^2y'^2\}. \qquad (11)$$

346. If in the preceding example the vertex of the triangle lies on one of the axes of the curve, show that the circle has double contact with a fixed concentric sphero-conic.

347. If the centre of the circle in Ex. 345 lies on the curve, show that the vertex of the triangle lies on a cone of the fourth order.

348. Suppose, in Ex. 345, that the vertex of the triangle lies on one of the great circles

$$a(b+c)(c-a)x'^2 - b(c+a)(b-c)y'^2 = 0, \qquad (1)$$

then the equation of the corresponding circle reduces to

$$\{xx'(c-a) - yy'(b-c)\}^2 - c\{(c-a)x'^2 + (c-b)y'^2\}(x^2+y^2+z^2) = 0,$$

and is, therefore, from (1), altogether fixed.

Writing one of these circles in the form

$$x\sqrt{b(c^2-a^2)} \pm y\sqrt{a(b^2-c^2)} = c\sqrt{\{(b-a)(x^2+y^2+z^2)\}},$$

we can easily see that it has double contact with the curve. We also see, as at Ex. 82, that the curve is its own reciprocal with regard to one of these circles.

There are besides two other pairs of such circles corresponding to triangles having their vertices on the two other axes of the curve.

349. To find the locus of the vertex of a triangle formed by two tangents to a sphero-conic and their chord of contact, if the centre of the inscribed circle lies on the curve. In the same way as at Ex. 85, we can show that if the curve is referred to the triangle, it must be written in the form

$$\alpha\beta - \gamma^2 = 0.$$

Putting, then,

$$x\cos\alpha + y\cos\beta + z\cos\gamma,\ x\cos\alpha' + y\cos\beta' + z\cos\gamma',$$
$$x\cos\lambda + y\cos\mu + z\cos\nu$$

for α, β, γ, respectively, in (1), it must become identical with

$$U \equiv ax^2 + by^2 + cz^2 = 0\ ;$$

hence if $\alpha\beta - \gamma^2 \equiv kU$, we have

$$\cos\alpha\cos\alpha' - \cos^2\lambda = ka,\ \&c., \qquad (1)$$

$$\cos\alpha\cos\beta' + \cos\alpha'\cos\beta = 2\cos\lambda\cos\mu,\ \&c.\ ; \qquad (2)$$

therefore $(\cos\alpha\cos\beta' - \cos\alpha'\cos\beta)^2 = -4k^2ab$

$$- 4k(a\cos^2\mu + b\cos^2\lambda). \qquad (3)$$

But $\cos\alpha\cos\beta' - \cos\alpha'\cos\beta = z\sin\theta,$

and, from (1), $k(a + b + c) = -(1 - \cos\theta)$;

also $\cos\lambda = \dfrac{ax}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}$ $\cos\mu = \&c.,$

where θ is the length of the base of the triangle.

Hence (3) becomes, after dividing by $1 - \cos\theta$,

$$p^2z^2(1 + \cos\theta) + 4ab(1 - \cos\theta)(x^2 + y^2 + z^2)$$
$$= \frac{4pab(ax^2 + by^2)}{a^2x^2 + b^2y^2 + c^2z^2}, \qquad (4)$$

and, by symmetry,

$$p^2y^2(1 + \cos\theta) + 4ac(1 - \cos\theta)(x^2 + y^2 + z^2)$$
$$= \frac{4pac(ax^2 + cz^2)}{a^2x^2 + b^2y^2 + c^2z^2}, \qquad (5)$$

where $p = a + b + c.$

Eliminating, then, $\cos\theta$ between (4) and (5), we get, after dividing by $by^2 - cz^2$,

$$p^2 U - 2p(a^2x^2 + b^2y^2 + c^2z^2) - 4abc(x^2 + y^2 + z^2) = 0,$$

or
$$\frac{ax^2}{b + c - a} + \frac{by^2}{a + c - b} + \frac{cz^2}{a + b - c} = 0,$$

which represents a sphero-conic confocal with U.

350. There are three systems of circles which have double contact with a sphero-conic, the chords of contact of each system passing through one of the centres of the curve.

The sphero-conic being written in the form

$$x^2 + y^2 + z^2 - 1 = 0, \quad ax^2 + by^2 + cz^2 = 0,$$

the equation
$$(c - a)xx' + (c - b)yy' - c = 0 \tag{1}$$

evidently represents a circle having double contact with the curve at points lying on the great circle $xy' - yx' = 0$. If we suppose xy to be the internal centre of the curve, the circles of the system (1) touch two opposite branches.

The equations

$$(b - a)xx' + (b - c)zz' - b = 0, \tag{2}$$

$$(a - b)yy' + (a - c)zz' - a = 0 \tag{3}$$

represent circles of the other two systems having double contact with the curve at points lying on the great circles

$$xz' - zx' = 0, \quad yz' - zy' = 0,$$

respectively. These circles touch the curve at points on the same branch.

If R is the radius of the circle (1), and ϕ the distance of its centre from the point xz, we find

$$\cos^2 R = \frac{c \cos^2 \phi}{c - a} + \frac{c \sin^2 \phi}{c - b}; \tag{4}$$

and for the circles (2) and (3) we get the relations

$$\cos^2 R = \frac{b \cos^2 \delta}{b - c} + \frac{b \sin^2 \delta}{b - a}, \tag{5}$$

$$\cos^2 R = \frac{a \cos^2 \delta'}{a - c} + \frac{a \sin^2 \delta'}{a - b}, \tag{6}$$

where δ, δ' are the distances of their centres from the internal centre of the curve.

Putting

$$a = - c \cot^2 \alpha, \quad b = - c \cot^2 \beta$$

in these equations, they become

$$\cos^2 R = \sin^2 \alpha \cos^2 \phi + \sin^2 \beta \sin^2 \phi, \tag{7}$$

$$\cos^2 R = \cos^2 \beta \cos^2 \delta + \frac{\cos^2 \beta \sin^2 \alpha}{\sin^2 \alpha - \sin^2 \beta} \sin^2 \delta, \tag{8}$$

$$\cos^2 R = \cos^2 \alpha \cos^2 \delta' - \frac{\cos^2 \alpha \sin^2 \beta}{\sin^2 \alpha - \sin^2 \beta} \sin^2 \delta'. \tag{9}$$

Hence we see from (7) that the radii of the circles of the system (1) vary between the limits $\frac{\pi}{2} - \alpha$, $\frac{\pi}{2} - \beta$. Also the maximum value of the radius of (2) is equal to β, and the minimum value of the radius of (3) is α. For the latter systems of circles the radii have their minimum and maximum values, respectively, when the circles have four-point contact with the curve at the vertices.

351. To find the differential equations in elliptic co-ordinates of the systems of circles having double contact with a sphero-conic.

Let the equations

$$\left.\begin{aligned}\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - b^2} - \frac{z^2}{c^2 - \mu^2} = 0,\\ \frac{x^2}{\nu^2} - \frac{y^2}{b^2 - \nu^2} - \frac{z^2}{c^2 - \nu^2} = 0,\end{aligned}\right\} \qquad (1)$$

combined with the equation of the sphere, determine two intersecting confocal sphero-conics; then we have

$$bcx = \mu\nu, \quad b\sqrt{(c^2 - b^2)}\,y = \sqrt{\{(\mu^2 - b^2)(b^2 - \nu^2)\}}$$

$$c\sqrt{(c^2 - b^2)}\,z = \sqrt{\{(c^2 - \mu^2)(c^2 - \nu^2)\}}. \qquad (2)$$

Now, if we have the differential equation

$$\frac{d\mu}{\sqrt{\{(\mu^2 - a^2)(\mu^2 - b^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2 - \nu^2)(b^2 - \nu^2)\}}} = 0, \qquad (3)$$

the integral of this equation, by the theory of elliptic functions, can be written in either of the forms

$$\left.\begin{aligned}A\mu\nu + B\sqrt{\{(\mu^2 - b^2)(b^2 - \nu^2)\}} + C = 0,\\ A'\mu\nu + B'\sqrt{\{(\mu^2 - a^2)(a^2 - \nu^2)\}} + C' = 0.\end{aligned}\right\} \qquad (4)$$

Hence, since $(\mu^2 - a^2)(a^2 - \nu^2)$ is proportional to the sphero-conic corresponding to the value $\mu = a = c\sin\alpha$, we see, from (4), that (3) is the differential equation of the system of circles (1) in the preceding example.

In the same way we can show that the differential equations of the other two systems of circles are

$$\frac{d\mu}{\sqrt{\{(a^2-\mu^2)(c^2-\mu^2)\}}} \pm \frac{d\nu}{\sqrt{(\{a^2-\nu^2)(c^2-\nu^2)\}}} = 0, \quad (5)$$

$$\frac{\mu d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-b^2)(c^2-\mu^2)\}}} \pm \frac{\nu d\nu}{\sqrt{\{(a^2-\nu^2)(b^2-\nu^2)(c^2-\nu^2)\}}} = 0. \quad (6)$$

From these equations it follows at once that two circles of the same system make equal angles with the two confocal conics which pass through their intersection.

352. To find the angle ϕ between two circles of the same system which have double contact with a sphero-conic.

If $d\sigma_1$, $d\sigma_2$ are the elements of the arcs of the two confocal sphero-conics, we have

$$d\sigma_1 = \frac{\sqrt{(\mu^2-\nu^2)}\,d\nu}{\sqrt{\{(b^2-\nu^2)(c^2-\nu^2)\}}}, \quad d\sigma_2 = \frac{\sqrt{(\mu^2-\nu^2)}\,d\mu}{\sqrt{\{(\mu^2-b^2)(c^2-\mu^2)\}}}; \quad (1)$$

hence, if a curve defined by the equation

$$Pd\mu + Qd\nu = 0$$

meet the curve μ at the angle i, we get

$$\tan i = \frac{Q}{P}\sqrt{\left\{\frac{(b^2-\nu^2)(c^2-\nu^2)}{(\mu^2-b^2)(c^2-\mu^2)}\right\}}. \quad (2)$$

But $i = \frac{1}{2}\phi$, and putting then for P, Q the values obtained from (3) in the preceding example, we get

$$\tan\tfrac{1}{2}\phi = \sqrt{\left\{\frac{(\mu^2-a^2)(c^2-\nu^2)}{(a^2-\nu^2)(c^2-\mu^2)}\right\}}. \quad (3)$$

From (3) we find

$$\cos\phi = \frac{(a^2+c^2)(\mu^2+\nu^2) - 2(a^2c^2+\mu^2\nu^2)}{(c^2-a^2)(\mu^2-\nu^2)};$$

but from (1) in the preceding example, we have

$$\mu^2 + \nu^2 = c^2 y^2 + b^2 z^2 + (b^2 + c^2) x^2.$$

Thus we see that if the circles cut orthogonally, the locus of their intersection is a sphero-conic having the same centres as the given one.

If the circles cut at a constant angle, their intersection will lie on a cone of the fourth order.

353. If a tangent great circle to the sphero-conic defined by the equation $\mu = a$ in elliptic co-ordinates make the angle i with the confocal at the point μ, ν, we can show that

$$\mu^2 \cos^2 i + \nu^2 \sin^2 i = a^2 ; \qquad (1)$$

hence, if such a great circle cut orthogonally the circle represented by the equation (3), Ex. 351, we have

$$1 + \tan i \tan i' = 0 ;$$

and, therefore, from (3) in the preceding example,

$$(\mu^2 - a^2) \sqrt{(c^2 - \nu^2)} - (a^2 - \nu^2) \sqrt{(a^2 - \mu^2)} = 0 ;$$

or, clearing of radicals and dividing by $\mu^2 - \nu^2$,

$$(c^2 - \mu^2)(c^2 - \nu^2) - (c^2 - a^2)^2 = 0,$$

which becomes, by transformation to Cartesian co-ordinates (see Ex. 351, (2)),

$$c^2 (c^2 - b^2) z^2 - (c^2 - a^2)^2 (x^2 + y^2 + z^2) = 0. \qquad (2)$$

Thus we see that one of the tangent great circles, drawn to the curve from any point of the small circle (2), passes through the centre of a circle of the system (1), Ex. 350, which passes through the same point.

354. If δ is the distance between the centres of two circles of the system (1), Ex. 350, we have

$$\{(\cos^2\alpha - \sin^2\beta)(x^2 + y^2) - (\sin^2\alpha + \sin^2\beta)z^2\}^2$$
$$- 4\sin^2\alpha\sin^2\beta\cot^2\delta\,(x^2\cot^2\alpha + y^2\cot^2\beta - z^2) = 0,$$

where x, y, z are the co-ordinates of the intersection of the circles. For the angle δ is evidently the angle between those tangents to the conic (see *Conics*, Art. 169, Ex. 3)

$$z = 0, \quad \frac{x^2}{\sin^2\alpha} + \frac{y^2}{\sin^2\beta} = 1,$$

which are the projections of the circles on the plane of xy.

Hence, if the centres of the circles are 90° distant from each other, their intersection will lie on the small circle

$$(\cos^2\alpha - \sin^2\beta)(x^2 + y^2) - (\sin^2\alpha + \sin^2\beta)z^2 = 0,$$

and their radii ρ, ρ' will be connected by the relation (Ex. 350, (7)),

$$\cos^2\rho + \cos^2\rho' = \sin^2\alpha + \sin^2\beta.$$

355. To find the locus of the points through which circles of the systems (1) and (3), Ex. 350, cut each other orthogonally.

From (3), Ex. 352, we have, for the first system,

$$\tan i = \sqrt{\left\{\frac{(\mu^2 - a^2)(c^2 - \nu^2)}{(a^2 - \nu^2)(c^2 - \mu^2)}\right\}}; \qquad (1)$$

and similarly for the second system from Ex. 351, (6),

$$\tan i' = \frac{\nu}{\mu}\sqrt{\left(\frac{\mu^2 - a^2}{a^2 - \nu^2}\right)}. \qquad (2)$$

But $$1 + \tan i \tan i' = 0;$$

hence, if we put $\mu = c\sin\phi, \quad \nu = c\sin\psi,$

we have, from (1) and (2),

$$\sin 2\phi(2a^2 - c^2 + c^2\cos 2\psi) \pm \sin 2\psi(2a^2 - c^2 + c^2\cos 2\phi) = 0,$$

which gives, after dividing by $\sin(\phi \pm \psi)$,

$$a^2\cos\phi\cos\psi \pm (c^2 - a^2)\sin\phi\sin\psi = 0;$$

or, transforming to Cartesian co-ordinates,

$$a^2\sqrt{(c^2 - b^2)}\,z \pm (c^2 - a^2)\,bx = 0,$$

which represents a pair of great circles passing through the centre xz.

356. If θ is the angle between the two circles considered in the preceding example, we find

$$\tan\theta = \frac{\sqrt{\{(a^2-b^2)(c^2-a^2)x^2 + a^2(c^2-a^2)y^2 - a^2(a^2-b^2)z^2\}}}{a^2\sqrt{(c^2-b^2)}z \pm (a^2-c^2)bx};$$

hence we see that if θ is given, the locus of the intersection of the circle consists of two sphero-conics having double contact with the given curve.

357. If θ is the angle between a tangent great circle to the sphero-conic $\mu = a$, and a circle of the system (3), Ex. 351, show that

$$\tan\theta = \frac{\sqrt{\{(a^2-b^2)(c^2-a^2)x^2 + a^2(c^2-a^2)y^2 - a^2(a^2-b^2)z^2\}}}{c^2 - a^2 \pm c\sqrt{(c^2-b^2)}z}.$$

358. If we take two circles of the system (2), Ex. 350,

their centres of similitude are determined by the equation

$$\frac{\sin(\phi - \delta)}{\sin(\phi - \delta')} = \pm \frac{\sin R}{\sin R'};$$

hence $$\tan\phi = \frac{\sin R \sin\delta' \mp \sin R' \sin\delta}{\sin R \cos\delta' \mp \sin R' \cos\delta},$$

and therefore

$$\tan\phi \tan\phi' = \frac{\sin^2 R \sin^2\delta' - \sin^2 R' \sin^2\delta}{\sin^2 R \cos^2\delta' - \sin^2 R' \cos^2\delta}. \qquad (1)$$

Now, (8), Ex. 350, can be written in the form

$$\frac{\sin^2 R}{\sin^2\beta} + \frac{\sin^2\delta}{\sin^2\gamma} = 1, \qquad (2)$$

where γ is the distance of a focus from the centre of the curve. Hence, from (1) and (2), we have

$$\tan\phi \tan\phi' = \tan^2\gamma,$$

which shows that the centres of similitude of two circles of the system considered are harmonically conjugate with the foci.

In the same way we can show that the centres of similitude of a pair of circles of either of the two other systems subtend a right angle at either of the foci.

359. From (2) in the preceding example we can show that if θ, θ' are the angles which the tangent great circles from the foci to the circles of the system (2), Ex. 350, make with the axis, then we have

$$\sin\theta \sin\theta' = \frac{\sin^2\beta}{\sin^2\gamma}.$$

360. To show that the circles of the system (1), Ex. 350, cut off constant intercepts from the cyclic arcs.

Let the equation of one of the circles be written

$$\frac{x}{\sin\alpha}\cos\phi + \frac{y}{\sin\beta}\sin\phi - 1 = 0, \qquad (1)$$

and that of a cyclic arc

$$y\sqrt{(\sin^2\alpha - \sin^2\beta)} - z\sin\beta = 0$$

(see Ex. 333); then we may put

$$y = \frac{\sin\beta}{\sin\alpha}\sin\rho, \quad z = \frac{\sqrt{(\sin^2\alpha - \sin^2\beta)}}{\sin\alpha}\sin\rho, \qquad (2)$$

where ρ is the distance of xyz from yz. Hence, since $x = \cos\rho$, from (1) we have

$$\cos\rho\cos\phi + \sin\rho\sin\phi = \sin\alpha,$$

and, therefore, $\rho - \phi = \frac{\pi}{2} - \alpha$, or $\alpha - \frac{\pi}{2}$,

from which we get $\rho_1 - \rho_2 = \pi - 2\alpha$.

361. To find the envelope of the radical axes of a fixed circle and the system of circles at (1), Ex. 350.

Let the equation of the variable circle be written

$$\left(\frac{x}{\sin\alpha}\cos\phi + \frac{y}{\sin\beta}\sin\phi\right)^2 - (x^2 + y^2 + z^2) = 0,$$

and that of the fixed circle

$$(lx + my + nz)^2 - (x^2 + y^2 + z^2) = 0; \qquad (1)$$

then for a radical axis we have

$$lx + my + nz \pm \left(\frac{x}{\sin\alpha}\cos\phi + \frac{y}{\sin\beta}\sin\phi\right) = 0; \qquad (2)$$

but the envelope of this great circle is

$$(lx + my + nz)^2 - \left(\frac{x^2}{\sin^2\alpha} + \frac{y^2}{\sin^2\beta}\right) = 0, \qquad (3)$$

which represents a sphero-conic touching the imaginary cyclic arcs

$$\frac{x^2}{\sin^2\alpha} + \frac{y^2}{\sin^2\beta} = 0,$$

and passing through the points where the fixed circle meets the given curve.

When the circle (1) is touched by a circle of the variable system, the radical axis is the tangent at the point of contact. Hence, to find the points where (1) is touched by four circles of the system, we draw the common tangents of (1) and (3), and then these tangents touch (1) in the required points.

362. Reciprocally we can show that the locus of the centres of similitude of a fixed circle S and a system of circles having double contact with a sphero-conic U is a sphero-conic touching the common tangents of S and U, and passing through a pair of real or imaginary foci of U.

This envelope will also determine the four points on S where it is touched by circles of the system.

363. A great circle touches two circles having double contact with a sphero-conic, to show that its points of contact with them lie on the same concyclic sphero-conic.

For the circles we have, from (3), Ex. 352,

$$\tan i = \sqrt{\left\{\frac{(\mu^2 - a^2)(c^2 - \nu^2)}{(a^2 - \nu^2)(c^2 - \mu^2)}\right\}};$$

and if the great circle touch the confocal conic $\mu = a'$, we have (Ex. 353, (1)),

$$\tan i' = \sqrt{\left(\frac{\mu^2 - a'^2}{a'^2 - \nu^2}\right)};$$

hence, since $i = i'$ in the case we are considering, we get

$$\frac{(\mu^2 - a'^2)(c^2 - \mu^2)}{\mu^2 - a^2} - \frac{(a'^2 - \nu^2)(c^2 - \nu^2)}{a^2 - \nu^2} = 0,$$

or, dividing by $\mu^2 - \nu^2$,

$$(\mu^2 - a^2)(a^2 - \nu^2) - (c^2 - a^2)(a^2 - a'^2) = 0, \qquad (1)$$

which, by transformation to Cartesian co-ordinates, gives the result stated above.

Putting $a' = c \sin\gamma$ in (1), we see that the points of the contact of the tangent great circles from the foci to the system lie on

$$(\mu^2 - a^2)(a^2 - \nu^2) - (c^2 - a^2)(a^2 - b^2) = 0,$$

which, being transformed to Cartesian co-ordinates, is found to represent the real cyclic arcs of the curve.

364. Through the centre of a circle having double contact with a sphero-conic tangent great circles are drawn to a confocal sphero-conic; show that they meet the circle on a small circle passing through the intersection of the given curves.

365. Tangents are drawn from a focus of the curve to circles of the system (1), Ex. 350; to show that they contain a constant angle.

If θ is the angle between the tangents, we have

$$\sin\tfrac{1}{2}\theta = \frac{\sin R}{\sin\Delta}, \qquad (1)$$

where Δ is the distance of the focus from the centre of the circle; but

$$\sin^2\Delta = 1 - \sin^2\gamma\cos^2\delta, \qquad (2)$$

and, from (7), Ex. 350,

$$\cos^2 R = \sin^2 a \cos^2 \delta + \sin^2 \beta \sin^2 \delta,$$

or

$$\sin^2 R = \cos^2 \beta (1 - \sin^2 \gamma \cos^2 \delta); \qquad (3)$$

hence, from (1), (2), and (3) we get

$$\theta = \pi - 2\beta.$$

This result might also be arrived at by reciprocating Ex. 360.

366. Show that a variable circle, having double contact with a sphero-conic, meets two fixed tangent great circles of the curve at angles whose sum or difference is constant.

Also show that a variable tangent great circle meets two fixed circles, having double contact with the curve, at angles whose sum or difference is constant (see Ex. 119).

367. A circle, having double contact with a sphero-conic, cuts orthogonally a circle having double contact with a confocal sphero-conic; if the circles have their centres on the same axis, show that the locus of their intersection is a sphero-conic having the same centres as the given ones.

368. If we are given a circle of the system (2), and another of the system (3), Ex. 350, we have the relations (8) and (9) in the same example, which we may write in the form

$$\tan^2 a \sin^2 R + \tan^2 \beta \cos^2 R = \frac{\sin^2 \beta}{\cos^2 a} \cos^2 \delta, \qquad (1)$$

$$\tan^2 a \cos^2 R' + \tan^2 \beta \sin^2 R' = \frac{\sin^2 a}{\cos^2 \beta} \cos^2 \delta'. \qquad (2)$$

Now if D is the distance between the centres of the circles, we have

$$\cos D = \cos \delta \cos \delta';$$

hence, from (1) and (2) we get

$$(\tan^2\alpha \sin^2 R + \tan^2\beta \cos^2 R)(\tan^2\alpha \cos^2 R' + \tan^2\beta \sin^2 R') = \tan^2\alpha \tan^2\beta \cos^2 D, \qquad (3)$$

from which we see that $\dfrac{\tan\alpha}{\tan\beta}$ has one or other of two given values.

Similarly, if we are given circles of the systems (1) and (3), Ex. 350, we can show that the axis major α of the curve is given; and given circles of the systems (1) and (2), β is given.

369. A circle, having double contact with a sphero-conic, touches a circle having double contact with a confocal sphero-conic; if the circles have their centres on different axes, show that the locus of the point of contact is a concentric sphero-conic.

If the circles cut each other orthogonally, show that the locus of their intersection consists of two chords of intersection of the given curves.

370. A circle whose radius is ρ and centre $x, 0, z$ cuts the sphero-conic

$$ax^2 + by^2 + cz^2 = 0$$

orthogonally at two points, show that

$$\tan^2\rho = \frac{(ax^2 + cz^2)}{(c - a)x^2 z^2}\left\{ax^2\left(1 - \frac{c}{b}\right) + cz^2\left(1 - \frac{a}{b}\right)\right\}.$$

Hence, show that the circle also cuts the sphero-conic

$$\frac{ax^2}{b - a} - y^2 + \frac{cz^2}{b - c} = 0$$

orthogonally at two points.

371. Show that the envelope of the circles considered in the preceding example is projected on the plane of xz into the evolute of the conic

$$\frac{x^2}{c^2(b-a)} + \frac{z^2}{a^2(b-c)} - \frac{b}{(ab+bc-ca)^2} = 0.$$

372. By the same method as that which we used in Ex. 162 we can show that if triangles be inscribed in the sphero-conic

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$

and circumscribed about the sphero-conic

$$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0,$$

then the normals to S at the vertices of the triangle pass through a point.

Also the sphero-conic

$$S'' \equiv (b-c)\,x'yz + (c-a)\,y'zx + (a-b)\,z'xy = 0$$

passes through the feet of the normals drawn from x', y', z' to S. Hence, since S'' circumscribes triangles circumscribed about S', we find from the invariant relation connecting the two latter conics that the points through which the normals pass lie on the sphero-conic

$$a'(b-c)^2x^2 + b'(c-a)^2 + c'(a-b)^2z^2 = 0. \qquad (1)$$

If $$\frac{a'}{a}(b-c)^2 = \frac{b'}{b}(c-a)^2 = \frac{c'}{c}(a-b)^2, \qquad (2)$$

the locus (1) coincides with S. Thus we see that if normals be drawn to the curve from any point on itself, the arcs joining their feet form a triangle circumscribed about the sphero-conic

$$\frac{ax^2}{(b-c)^2} + \frac{by^2}{(c-a)^2} + \frac{cz^2}{(a-b)^2} = 0.$$

We are permitted to assume the equations (2), for they are consistent with

$$\sqrt{\left(\frac{a}{a'}\right)}+\sqrt{\left(\frac{b}{b'}\right)}+\sqrt{\left(\frac{c}{c'}\right)}=0,$$

the invariant relation connecting S and S'.

373. From the point where a normal to a sphero-conic touches the evolute two other normals are drawn to the curve; show that the great circle joining their feet is a normal to a concentric sphero-conic.

374. In the same way as in Ex. 204 we can show that if a sphero-conic have double contact with a fixed sphero-conic U and a fixed circle V, then a pair of foci will lie on one of the conics confocal with U, which pass through the extremities of a diagonal of the quadrilateral formed by the common tangents of U and V.

These pair of foci lie on a great circle passing through the centre of V; and if they are imaginary, the real foci will be the anti-points of a pair of points in which a great circle passing through a fixed point meets a fixed sphero-conic. To find this locus, let the fixed point be x', y', z', and the fixed sphero-conic

$$S \equiv ax^2 + by^2 + cz^2 = 0.$$

Forming, then, the equation of the chords of intersection of S and

$$PQ \equiv (x'^2 + y'^2 + z'^2)(x^2 + y^2 + z^2) - (xx' + yy' + zz')^2 = 0\ ;$$

and expressing that this equation is satisfied by the co-ordinates of the fixed point, we obtain the locus required:

$$abc\, P^2 Q^2 - S' PQF + S'^2 S\,(x^2 + y^2 + z^2) = 0,$$

where

$$S' = ax'^2 + by'^2 + cz'^2, \quad F = a(b + c)\,x^2 + b(c + a)\,y^2 + c(a + b)\,z^2.$$

Thus the complete locus for the three systems of variable curves consists of three confocal sphero-conics and three curves lying on cones of the fourth order.

375. A sphero-conic has double contact with a fixed circle and touches two great circles; show that the locus of its foci consists of two great circles and a curve lying on a cone of the fourth order.

376. If μ and ν have the same meaning as in (1), Ex. 339, show that the differential equation

$$\frac{d\mu}{\sqrt{\{(\mu^2-a^2)(\mu^2-a'^2)(c^2-\mu^2)\}}} \pm \frac{d\nu}{\sqrt{\{(a^2-\nu^2)(a'^2-\nu^2)(c^2-\nu^2)\}}} = 0$$

represents one of the systems of sphero-conics which have double contact with the confocal sphero-conics $\mu = a$, $\mu = a'$ (see Ex. 212).

377. To show that the sum of the cotangents of the common tangents of a circle and a sphero-conic is equal to zero.

Let p be the length of the perpendicular from a fixed point on a tangent to the curve, and let ω be the angle which the perpendicular makes with a fixed line; then we know that

$$\cot t = \cos p \, \frac{d\omega}{dp}, \qquad (1)$$

where t is the length of the intercept between the point of contact and the foot of the perpendicular.

Hence, since p is the same for the four tangents which touch a circle having the fixed point for centre, we have, from (1),

$$\Sigma \cot t = \cos p \, \frac{d\,\Sigma\omega}{dp}. \qquad (2)$$

But projecting the circle and sphero-conic by rays through the centre of the sphere on the tangent plane at the fixed point, these curves are transformed into a circle and a conic, respectively, and the angle ω becomes the corresponding angle ω in the plane. Now we have seen in Ex. 232 that $\Sigma\omega$ in the plane is independent of p; hence, from (2), we obtain

$$\Sigma \cot t = 0.$$

378. Show that a circle meets a sphero-conic at angles the sum of whose co-tangents is equal to zero.

379. If xyz, $x'y'z'$ are the co-ordinates of the centres, and ρ, ρ' the radii of the polar and circumscribing circles, respectively, of a triangle self-conjugate with regard to the sphero-conic

$$ax^2 + by^2 + cz^2 = 0,$$

show that

$$(axx' + byy' + czz')^2 - \sin^2\rho(a^2x'^2 + b^2y'^2 + c^2z'^2)$$
$$- \sin^2\rho'(a^2x^2 + b^2y^2 + c^2z^2) + \sin^2\rho \sin^2\rho'(a^2 + b^2 + c^2) = 0$$

(see Ex. 273).

380. Given four tangents to a sphero-conic, to find the locus of the foci.

Let α, β, γ, δ be the sines of the perpendiculars from a point on the four tangents, and let

$$l\alpha + m\beta + n\gamma + p\delta = 0 \qquad (1)$$

be an identical relation connecting these perpendiculars. Then, since the product of the sines of the perpendiculars from the foci on any tangent is constant, if one focus satisfy (1), the other must lie on

$$l\beta\gamma\delta + m\gamma\delta\alpha + n\delta\alpha\beta + p\alpha\beta\gamma = 0,$$

which represents a cubic cone passing through all the intersections of the given tangents.

381. If the two sphero-conics described through a point P to touch four fixed great circles cut orthogonally at P, show that P lies on the locus found in the preceding example (see Ex. 298).

382. Given four points on a sphero-conic, show that the locus of the foci is

$$l \sin \alpha + m \sin \beta + n \sin \gamma + p \sin \delta = 0,$$

where α, β, γ, δ are the distances of a point from the four given points, and l, m, n, p are such that

$$l \cos \alpha + m \cos \beta + n \cos \gamma + p \cos \delta = 0,$$

identically.

383. Right-angled triangles are inscribed in a sphero-conic; show that the locus of the point where the normal at the right angle meets the opposite side is a sphero-conic having its centres in common with the given curve.

Also, if the curve is equilateral (see Ex. 344), show that the intersection of the normal and the opposite side is the pole of the tangent.

384. To determine the direction of the two chords of a sphero-conic which may be drawn through a fixed point in space.

Let x', y', z' be the co-ordinates of the fixed point, and let us take

$$x = x' + \rho \cos \alpha, \quad y = y' + \rho \cos \beta, \quad z = z' + \rho \cos \gamma$$

oiy faopnrnt on a line passing through $x'y'z'$. Substituting,